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To be or not to be constructive That is not the question

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Abstract

In the early twentieth century, L.E.J. Brouwer pioneered a new philosophy of mathematics, called *intuitionism*. Intuitionism was revolutionary in many respects but stands out – mathematically speaking – for its challenge of *Hilbert's formalist philosophy* of mathematics and rejection of the *law of excluded middle* from the 'classical' logic used in mainstream mathematics. Out of intuitionism grew *intuitionistic logic* and the associated *Brouwer–Heyting–Kolmogorov* interpretation by which 'there exists x' intuitively means 'an algorithm to compute x is given'. A number of schools of constructive mathematics were developed, inspired by Brouwer's intuitionism and invariably based on intuitionistic logic, but with varying interpretations of what constitutes an algorithm. This paper deals with the *dichotomy* between constructive and non-constructive mathematics, or rather the *absence* of such an 'excluded middle'. In particular, we challenge the 'binary' view that mathematics is either constructive or not. To this end, we identify a part of *classical* mathematics, namely *classical Nonstandard Analysis*, and show it inhabits the twilight-zone between the constructive and non-constructive. Intuitively, the predicate 'x is standard' typical of Nonstandard Analysis can be interpreted as 'x is computable', giving rise to computable (and sometimes constructive) mathematics obtained directly from *classical* Nonstandard Analysis. Our results formalise Osswald's longstanding conjecture that classical Nonstandard Analysis is *locally constructive*. Finally, an alternative explanation of our results is provided by Brouwer's thesis that *logic depends upon mathematics*.

1. Introduction

This volume is dedicated to the founder of *intuitionism*, L.E.J. Brouwer, who pursued this revolutionary programme with great passion and against his time's received view of mathematics

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and its foundations [16,18,41,109,111]. We therefore find it fitting that our paper attempts to subvert (part of) *our time's* received view of mathematics and its foundations. As suggested by the title, we wish to challenge the binary distinction *constructive*[1] *versus non-constructive* mathematics. We shall assume basic familiarity with constructive mathematics and *intuitionistic logic* with its *Brouwer–Heyting–Kolmogorov* interpretation.

Surprising as this may be to the outsider, the quest for the (ultimate) foundations of mathematics was and is an *ongoing and often highly emotional affair*. The *Grundlagenstreit* between Hilbert and Brouwer is perhaps the textbook example (see e.g. [111, II.13]) of a fierce struggle between competing views on the foundations of mathematics, namely Hilbert's formalism and Brouwer's intuitionism. Einstein was apparently disturbed by this controversy and exclaimed the following:

> What is this frog and mouse battle among the mathematicians? ([29, p. 133])

More recently, Bishop mercilessly attacked *Nonstandard Analysis* in his review [12] of Keisler's monograph [54], even going as far as debasing Nonstandard Analysis[2] to a *debasement of meaning* in [11]. Bishop, as Brouwer, believed that to state the existence of an object, one has to provide a construction for it, while Nonstandard Analysis cheerfully includes ideal/non-constructive objects *at the fundamental level*, the textbook example being infinitesimals. Note that Brouwer's student, the intuitionist Arend Heyting, had a higher opinion of Nonstandard Analysis [42].

A lot of ink has been spilt over the aforementioned struggles, and we do not wish to add to that literature. By contrast, the previous paragraph is merely meant to establish the well-known juxtaposition of *classical/mainstream/non-constructive* versus *constructive* mathematics.[3] The following quote by Bishop emphasises this 'two poles' view for the specific case of Nonstandard Analysis, which Bishop believed to be the worst exponent of classical mathematics.

> [Constructive mathematics and Nonstandard Analysis] are at opposite poles. Constructivism is an attempt to deepen the meaning of mathematics; non-standard analysis, an attempt to dilute it further. ([10, p. 1–2]).

To be absolutely clear, lest we be misunderstood, we only wish to point out the current state-of-affairs in contemporary mathematics: On one hand, there is mainstream mathematics with its classical logic and other fundamentally non-constructive features, of which Nonstandard Analysis is the *nec plus ultra* according to some; on the other hand, there is constructive mathematics with its intuitionistic logic and computational-content-by-design. In short, there are two opposing camps (classical and constructive) in mathematics separated by a no-man's land, with the occasional volley exchanged as in e.g. [4,78,95,96].

Stimulated by Brouwer's revolutionary spirit, our goal is to subvert the above received view. To this end, we will identify a field of classical mathematics which occupies the twilight-zone between constructive and classical mathematics. Perhaps ironically, this very field is

[1] The noun 'constructive' is often used as a synonym for 'effective', while it refers to the foundational framework *constructive mathematics* in logic and the foundations of mathematics [104]. Context determines the meaning of 'constructive' in this paper (usually the latter).

[2] The naked noun *Nonstandard Analysis* will always implicitly include the adjective *classical*, i.e. based on classical logic. We shall not directly deal with *constructive Nonstandard Analysis*, i.e. based on intuitionistic logic, but do discuss its relationship with our results in Section 5.3.

[3] A number of approaches to constructive mathematics exist ([5, III], [104, I.4], [13]), and both Brouwer's *intuitionism* and Bishop's *Constructive Analysis* [8] represent a school therein.

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