Available online at www.sciencedirect.com
ScienceDirect
Indagationes Mathematicae ▮ (▮▮▮▮) ▮▮▮–▮▮▮
indagationes mathematicae
www.elsevier.com/locate/indag

Virtual Special Issue — L.E.J. Brouwer after 50 years

On the cofinality of the splitting number

Alan Dow[a,*], Saharon Shelah[b,1]

[a] Department of Mathematics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA
[b] Department of Mathematics, Rutgers University, Hill Center, Piscataway, NJ, 08854-8019, USA

Abstract

The splitting number $\mathfrak{s}$ can be singular. The key method is to construct a forcing poset with finite support matrix iterations of ccc posets introduced by Blass and Shelah (1989).

Keywords: Splitting number; Cardinal invariants of the continuum; Matrix forcing

1. Introduction

The cardinal invariants of the continuum discussed in this article are very well known (see [4, van Douwen, p 111]) so we just give a brief reminder. They deal with the mod finite ordering of the infinite subsets of the integers. A set $S \subset \omega$ is *unsplit* by a family $\mathcal{Y} \subset [\omega]^{\aleph_0}$ if S is mod finite contained in one member of $\{Y, \omega \setminus Y\}$ for each $Y \in \mathcal{Y}$. The splitting number $\mathfrak{s}$ is the minimum cardinal of a family $\mathcal{Y}$ for which there is no infinite set unsplit by $\mathcal{Y}$ (equivalently every $S \in [\omega]^{\aleph_0}$ is *split* by some member of $\mathcal{Y}$). It is mentioned in [2] that it is currently unknown if $\mathfrak{s}$ can be a singular cardinal.

Proposition 1.1. *The cofinality of the splitting number is not countable.*

Proof. Assume that θ is the supremum of $\{\kappa_n : n \in \omega\}$ and that there is no splitting family of cardinality less than θ. Let $\mathcal{Y} = \{Y_\alpha : \alpha < \theta\}$ be a family of subsets of ω. Let $S_0 = \omega$ and by induction on n, choose an infinite subset S_{n+1} of S_n so that S_{n+1} is not split by the family

* Corresponding author.
E-mail addresses: adow@uncc.edu (A. Dow), shelah@math.rutgers.edu (S. Shelah).
[1] Current address: Institute of Mathematics, Hebrew University, Givat Ram, Jerusalem 91904, Israel.

http://dx.doi.org/10.1016/j.indag.2017.01.010
0019-3577/

Please cite this article in press as: A. Dow, S. Shelah, On the cofinality of the splitting number, Indagationes Mathematicae (2017), http://dx.doi.org/10.1016/j.indag.2017.01.010.

$\{Y_\alpha : \alpha < \kappa_n\}$. If S is any pseudointersection of $\{S_n : n \in \omega\}$, then S is not split by any member of $\mathcal{Y}$. $\square$

One can easily generalize the previous result and proof to show that the cofinality of the splitting number is at least $\mathfrak{t}$. In this paper we prove the following.

Theorem 1.2. *If κ is any uncountable regular cardinal, then there is a $\lambda > \kappa$ with $\mathrm{cf}(\lambda) = \kappa$ and a ccc forcing $\mathbb{P}$ satisfying that $\mathfrak{s} = \lambda$ in the forcing extension.*

To prove the theorem, we construct $\mathbb{P}$ using matrix iterations.

2. A special splitting family

Definition 2.1. Let us say that a family $\{x_i : i \in I\} \subset [\omega]^\omega$ is θ-Luzin (for an uncountable cardinal θ) if for each $J \in [I]^\theta$, $\bigcap\{x_i : i \in J\}$ is finite and $\bigcup\{x_i : i \in J\}$ is cofinite.

Clearly a family is θ-Luzin if every θ-sized subfamily is θ-Luzin. We leave to the reader the easy verification that for a regular uncountable cardinal θ, each θ-Luzin family is a splitting family. A poset being θ-Luzin preserving will have the obvious meaning. For example, any poset of cardinality less than a regular cardinal θ is θ-Luzin preserving.

Lemma 2.2. *If θ is a regular uncountable cardinal then any ccc finite support iteration of θ-Luzin preserving posets is again θ-Luzin preserving.*

Proof. We prove this by induction on the length of the iteration. Fix any θ-Luzin family $\{x_i : i \in I\}$ and let $\langle\langle \mathbb{P}_\alpha : \alpha \le \gamma\rangle, \langle \dot{\mathbb{Q}}_\alpha : \alpha < \gamma\rangle\rangle$ be a finite support iteration of ccc posets satisfying that $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha$ is ccc and θ-Luzin preserving, for all $\alpha < \gamma$. If γ is a successor ordinal $\beta + 1$, then for any $\mathbb{P}_\beta$-generic filter G_β, the family $\{x_i : i \in I\}$ is a θ-Luzin family in $V[G_\beta]$. By the hypothesis on $\dot{\mathbb{Q}}_\beta$, this family remains θ-Luzin after further forcing by $\dot{\mathbb{Q}}_\beta$.

Now we assume that α is a limit. Let $\dot{J}_0$ be any $\mathbb{P}_\gamma$-name of a subset of I and assume that $p \in \mathbb{P}_\gamma$ forces that $|\dot{J}_0| = \theta$. We must produce a $q < p$ that forces that $\dot{J}_0$ is as in the definition of θ-Luzin. There is a set $J_1 \subset I$ of cardinality θ satisfying that, for each $i \in J_1$, there is a $p_i < p$ with $p_i \Vdash i \in \dot{J}_0$. The case when the cofinality of α not equal to θ is almost immediate. There is a $\beta < \alpha$ such that $J_2 = \{i \in J_1 : p_i \in \mathbb{P}_\beta\}$ has cardinality θ. There is a $\mathbb{P}_\beta$-generic filter G_β such that $J_3 = \{i \in J_2 : p_i \in G_\beta\}$ has cardinality θ. By the induction hypothesis, the family $\{x_i : i \in I\}$ is θ-Luzin in $V[G_\beta]$ and so we have that $\bigcap\{x_i : i \in J_3\}$ is finite and $\bigcup\{x_i : i \in J_3\}$ is co-finite. Choose any $q < p$ in G_β and a name $\dot{J}_3$ for J_3 so that q forces this property for $\dot{J}_3$. Since q forces that $\dot{J}_3 \subset \dot{J}_0$, we have that q forces the same property for $\dot{J}_0$.

Finally we assume that α has cofinality θ. Naturally we may assume that the collection $\{\mathrm{dom}(p_i) : i \in J_1\}$ forms a Δ-system with root contained in some $\beta < \alpha$. Again, we may choose a $\mathbb{P}_\beta$-generic filter G_β satisfying that $J_2 = \{i \in J_1 : p_i \restriction \beta \in G_\beta\}$ has cardinality θ. In $V[G_\beta]$, let $\{J_{2,\xi} : \xi \in \omega_1\}$ be a partition of J_2 into pieces of size θ. For each $\xi \in \omega_1$, apply the induction hypothesis in the model $V[G_\beta]$, and so we have that $\bigcap\{x_i : i \in J_{2,\xi}\}$ is finite and $\bigcup\{x_i : i \in J_{2,\xi}\}$ is co-finite. For each $\xi \in \omega_1$ let m_ξ be an integer large enough so that $\bigcap\{x_i : i \in J_{2,\xi}\} \subset m_\xi$ and $\bigcup\{x_i : i \in J_{2,\xi}\} \supset \omega \setminus m_\xi$. Let m be any integer such that $m_\xi = m$ for uncountably many ξ. Choose any condition $\bar{p} \in \mathbb{P}_\alpha$ so that $\bar{p} \restriction \beta \in G_\beta$. We prove that for each $n > m$ there is a $\bar{p}_n < \bar{p}$ so that $\bar{p}_n \Vdash n \notin \bigcap\{x_i : i \in \dot{I}\}$ and $\bar{p}_n \Vdash n \in \bigcup\{x_i : i \in \dot{I}\}$. Choose any $\xi \in \omega_1$ so that $m_\xi = m$ and $\mathrm{dom}(p_i) \cap \mathrm{dom}(\bar{p}) \subset \beta$ for all $i \in J_{2,\xi}$. Now choose any $i_0 \in J_{2,\xi}$ so that $n \notin x_{i_0}$. Next choose a distinct ξ' with $m_{\xi'} = m$ so that $\mathrm{dom}(p_i) \cap (\mathrm{dom}(\bar{p}) \cup \mathrm{dom}(p_{i_0})) \subset \beta$ for

Please cite this article in press as: A. Dow, S. Shelah, On the cofinality of the splitting number, Indagationes Mathematicae (2017), http://dx.doi.org/10.1016/j.indag.2017.01.010.