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Dimension of compact metric spaces

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Abstract

We give a survey of old and new results in dimension theory of compact metric spaces. Most of the relatively new results presented in the survey are based on the cohomological dimension approach. We complement the survey by stating the basics of cohomological dimension theory and listing some of its applications beyond the dimension theory.

Keywords: Dimension; Cohomological dimension

1. Introduction

In 1911 Brouwer [12] proved that the m-cube I^m and the n-cube I^n are not homeomorphic for $m \neq n$. In the proof he used the property of the n-cube that it does not admit an ϵ-map for sufficiently small $\epsilon > 0$ to an $(n-1)$-dimensional polyhedron. This property leads to a concept of dimension of compact metric spaces which coincides with the modern definition. Formally this property was adopted as the definition of dimension by Alexandroff [1] in 1930.

We note that dimension of compact metric spaces admits many non-obviously equal definitions. The first formal definition was given by Brouwer [13] in 1913 though informally the ideas of such definition were proposed by Poincare [77]. Brouwer called his dimension invariant Dimensionsgrad and denoted it as Dg. He proved that $Dg(I^n) = n$. Later when other definitions of dimension appeared and were widely accepted Brouwer believed [14–17], that for compact metric spaces they coincide with his Dg. It turns out to be that he was right but the proof of it appeared almost a century later in a work of Fedorchuk, Levin and Shchepin [50].

Other famous contributers to the formulation of the concept of dimension are Lebesgue, Urysohn, Menger, Hurewicz and Čech.

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One of the pillars of dimension theory is the celebrated Brouwer Fixed Point theorem. Without the Brouwer Fixed Point theorem used explicitly or implicitly mere existence of topological spaces of dimension $n > 1$ cannot be established. We illustrate it on the following definition of dimension (Alexandroff [2]): $\dim X \geq n$ *if and only if* X *admits an essential map onto the* n*-cube.* We note again that for compact metric spaces all known definitions of dimension are equivalent. We recall that a continuous map $f : X \to I^n$ is *essential* if there is no continuous deformation fixed on the preimage $f^{-1}(\partial I^n)$ of the map f to a map to the boundary $g : X \to \partial I^n$. The Brouwer Fixed Point theorem is equivalent to the fact that there is no retraction of I^n onto the boundary ∂I^n. Thus, it implies that the identity map $id : I^n \to I^n$ is essential and, hence, $\dim I^n \geq n$.

In this survey we consider only compact metric spaces (called compacta) since many deep results in dimension theory are based on the cohomological dimension theory and the latter is developed to full extend for compacta only. Still the cohomological dimension theory works for σ-compact spaces but for general separable metric spaces it has different features and is not completely developed yet [46,47,62].

2. Equivalent definitions

Throughout this paper we assume that maps are continuous and spaces are separable metrizable. Most of the time they will be compact or locally compact. We recall that a *compactum* means a compact metric space. By dimension of a space $\dim X$ we assume the covering dimension.

Definition 2.1. The *covering dimension* of a topological space X does not exceed n, $\dim X \leq n$, if for every open cover $\mathcal{U}$ of X there is an open refinement $\mathcal{V} \prec \mathcal{U}$ of multiplicity $\leq n + 1$. We say that $\dim X = n$ if $\dim X \leq n$ and the condition $\dim X \leq n - 1$ does not hold true.

We recall that the refinement $\mathcal{V} \prec \mathcal{U}$ means that for each $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ with $V \subset U$. The multiplicity of a cover $\mathcal{V}$ is the maximal number of elements of $\mathcal{V}$ having a common point.

A family $\mathcal{U}$ of subsets of a metric space X is called an ϵ*-family* if the diameter of each set in the family does not exceed ϵ. An ϵ-family which is a cover of X is called an ϵ*-cover* of X.

Theorem 2.2. *For a compact metric space* X *the following are equivalent*

- (1) $\dim X \leq n$;
- (2) *Given* $\epsilon > 0$, *there is an open* ϵ*-cover* $\mathcal{U} = \mathcal{U}^0 \cup \cdots \cup \mathcal{U}^n$ *of* X *such that each family* $\mathcal{U}^i$ *consists of disjoint open sets;*
- (3) *For every* $\epsilon > 0$, X *admits an* ϵ*-map to an* n*-dimensional simplicial complex;*
- (4) *For any closed subset* $A \subset X$ *and a continuous map* $f : A \to S^n$ *there is a continuous extension* $\bar{f} : X \to S^n$;
- (5) X *does not admit an essential map onto* I^{n+1};
- (6) X *is the union of* $n + 1$ *subsets of dimension* 0;
- (7) X *admits a light map to* I^n;
- (8) X *is the image of a 0-dimensional compactum* Y *under a map* $f : Y \to X$ *with* $|f^{-1}(x)| \leq n + 1$ *for all* $x \in X$;
- (9) X *is the limit of an inverse system of* n*-dimensional polyhedra.*

By definition, ϵ-map $f : X \to Y$ is a map with $diam f^{-1}(y) \leq \epsilon$ for all $y \in Y$. A map with 0-dimensional point preimages is called *light*.

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