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Which linear operators preserve outer functions?

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Abstract

The aim of this study is to determine the systems that preserve the minimum phase property in signal processing. The minimum phase signals are closely related to outer functions. A basic mathematical question that arises in geophysical imaging is to characterize the linear operators preserving the set of outer functions in Hardy spaces. It is shown that a bounded linear operator on the Hardy space H^p, $1 < p < \infty$, preserving the set of outer functions is necessarily a weighted composition operator. Moreover, an operator preserving the set of shifted outer functions is necessarily a weighted composition operator as well. These results complement work by Gibson and Lamoureux.

Keywords: Hardy space; Outer function; Weighted composition operator

1. Introduction

This study is motivated by a physical question, namely, the characterization of the bounded linear operators preserving the impulsive nature of a gunshot report. In this regard, the class of shifted outer functions expresses delayed, causal, and minimum-phase signals that model impulsive physical sources such as gunshot reports. The aim of this study is to investigate the preservation of the minimum phase property in Hardy spaces [13,15], that is, to determine the

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operators in the system that maintain the set of all minimum phase signals. Interestingly enough, this geophysical problem can be transformed into the following basic mathematical problem.

Outer Preserving Problem *Which bounded linear operators preserve the set of outer functions in certain spaces of analytic functions?*

More precisely, let X be a Banach space and S a subset of X. A bounded linear operator T on X is said to preserve the set S if $TS \subset S$. That is, $Tf \in S$ whenever $f \in S$. Moreover, if X is a particular space of analytic functions, for instance, a Hardy space, and S is the set of outer functions on X, then T is called an *outer preserving operator* if $Tf \in S$ whenever $f \in S$.

Outer functions are important in the theory of analytic functions, particularly in the canonical factorization of functions in Hardy spaces. They can be seen as generalizations of stable polynomials, which have no zeros in the unit disc [3,14]. Recently, a close relationship between outer functions and minimum phase signals has been discovered [7,18]. In certain physical processes, minimum phase signals may be defined as front loaded signals whose energy is concentrated at the front [18]. The **Outer Preserving Problem** on the Hardy space $H^2(\mathbb{D})$ in the unit disc $\mathbb{D}$ was investigated in [15]. In particular, it was shown that a linear operator preserving the set of shifted outer functions is necessarily a weighted composition operator. Moreover, an application to seismic imaging was presented. Based on the results in [15], the linear operators preserving the set of minimum phase functions were characterized in [13]. Gibson et al. [14] determined all types of preserving operators on the Hardy space $H^2(\mathbb{D})$ using the structural theorem.

The aim of this study is to investigate the **Outer Preserving Problem** on the Hardy space $H^p(\mathbb{D})$ for $1 < p < \infty$. The results generalize those in [15] for $p \neq 2$. The paper is organized as follows. In Section 2, the background material on weighted composition operators is presented. In Section 3, the **Outer Preserving Problem** on $H^p(\mathbb{D})$, $(1 < p < \infty)$, is studied. In Section 4, it is shown that an operator preserving the set of shifted outer functions is a weighted composition operator. An application is presented at the end of the article.

2. Weighted composition operators

Let $\mathbb{C}$, $\mathbb{D}$, and $H(\mathbb{D})$ be the complex plane, the open unit disc, and the space of all analytic functions on $\mathbb{D}$, respectively. For $1 \le p < \infty$, the Hardy space $H^p(\mathbb{D})$ is the set of functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Moreover, let $H^\infty(\mathbb{D})$ be the space of functions in $H(\mathbb{D})$ satisfying

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)| < \infty.$$

For any $p \ge 1$, $H^p(\mathbb{D})$, with the norms defined above, is a Banach space and $H^p(\mathbb{D}) \subset H^1(\mathbb{D})$.

In 1960, De Leeuw et al. [8] characterized the isometry operators on $H^1(\mathbb{D})$. They are necessarily the weighted composition operators. Forelli [11] later showed that the same is true for $H^p(\mathbb{D})$ for any $p \neq 2$. The theory of weighted composition operators is currently a well-developed and active area of research in function theory [17,20]. Recently, Contreras et al. [4] studied weighted composition operators on Hardy spaces, whereas Cučković et al. [6] investigated bounded weighted composition operators between Bergman spaces and between Hardy spaces. In this section, we characterize the weighted composition operators preserving the set of outer functions.