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Disjointly improjective operators and domination problem

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Abstract

In this work we introduce the disjointly improjective operators between Banach lattices. We investigate this class of operators. Also, we extend the Flores–Hernández's theorem on the domination problem by disjoint strictly singular operator.

Keywords: Banach lattice; Positive operator; Order continuous norm; Strictly singular operator

1. Introduction and preliminary results

Given two operators $0 \leq R \leq T : E \to F$ between Banach lattices, that is, $0 \leq R(x) \leq T(x)$ in F for all positive $x \in E$, it is natural to ask whether the compactness of T would imply that of R. A positive answer was given by P.G. Dodds and D.H. Fremlin in [5] provided that E' and F both have order continuous norms. For weak compactness, A.W. Wickstead [21] proved that if T is weakly compact, and E' or F is order continuous, then R is also weakly compact. Along this line, some interesting classes of operators were considered in [7,12,13,20,3,4], and many other papers. In particular, for disjointly strictly singular operators, the main result of J. Flores and F.L. Hernández ([6], Theorem 1.1) asserts that if F has order continuous norm and T is disjointly-strictly singular, then so is R. Recall that an operator $T : E \to F$ is said to be disjointly strictly singular (DSS) if there is no disjoint sequence (x_n) of non null vectors in E

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such that the restriction of T to $[x_n]$ (the closed subspace spanned by the vectors (x_n)) is an isomorphism. DSS operators were first introduced by F. L. Hernández and B. Rodríguez-Salinas, in [11].

In the present paper, motivated by the domination problem, we introduce a new class of operators, called *disjointly improjective*, that in general properly contains the class of DSS operators. Our definition is based on the notion of improjective operators introduced by E. Tarafdar in [18] and [19].

Definition 1.1. An operator T from a Banach lattice E into a Banach space X is said to be disjointly improjective (DImp) if, there is no disjoint sequence (x_n) such that the restriction $T_{|[x_n]}$ is an isomorphism and $[T(|x_n|)]$ is complemented in X.

Note that the class of DImp operators includes DSS operators. As the following example shows, this inclusion is proper. In fact, the identity operator $I: l_\infty \to l_\infty$ is not DSS but DImp. Indeed, if there exists a disjoint complemented sequence (w_n) in l_∞, then by ([14], Theorem 2.a.7), $[w_n]$ is isomorphic to l_∞. Contradiction with the fact that l_∞ is not separable. In contrast when X is a disjointly complemented Banach lattice [8] then both classes coincide (see Theorem 2.4).

In Section 3, we extend ([6], Theorem 1.1) realizing the main purpose of this text. In the first stage, we prove that without the requirement that F has order continuous norm, if T is DSS then R is DImp (see Theorem 3.1). In the second stage, we prove that if F has order continuous norm and T is DImp, then so is R (see Theorem 3.2).

2. Some results on DImp operators

We recall from [15] the following theorem:

Theorem 2.1. *([15], Theorem 1.c.5) The following properties are equivalent for every infinite dimensional subspace X of an order continuous Banach lattice E.*

1. *X is reflexive.*
2. *No subspace of X is isomorphic to l_1 or to c_0.*

Definition 2.1 (*[10]*)**.** Let X and Y be two Banach spaces. An operator $T \in L(X, Y)$ is said to be almost weakly compact (AWC) if, whenever T has a bounded inverse on a closed subspace, M of X, then M is reflexive.

We start the investigation by proving the following result.

Theorem 2.2. *Let E and F be two Banach lattices with F having order continuous norm. Then each positive DImp operator from E to F is AWC.*

Proof. If T is not AWC, then there exists a non-reflexive infinite-dimensional subspace X of E such that $T_{|X}$ is an isomorphism. Since F is order continuous and X is not reflexive it follows from Theorem 2.1 that X contains a subspace isomorphic to c_0 or to l_1. Then by ([17], Theorems 3.4.14, 3.4.11 and Corollary 3.4.17) and since F has order continuous norm, there exists a normalized disjoint positive sequence (w_n) of E such that $[w_n]$ is isomorphic to l_1 or to c_0 and $T_{|[w_n]}$ is an isomorphism.