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Assignments for topological group actions

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Received 27 February 2016; received in revised form 18 April 2017; accepted 5 September 2017

Communicated by: M.N. Crainic

Abstract

A polynomial assignment for a continuous action of a compact torus T on a topological space X assigns to each $p \in X$ a polynomial function on the Lie algebra of the isotropy group at p in such a way that a certain compatibility condition is satisfied. The space $\mathcal{A}_T(X)$ of all polynomial assignments has a natural structure of an algebra over the polynomial ring of Lie(T). It is an equivariant homotopy invariant, canonically related to the equivariant cohomology algebra. In this paper we prove various properties of $\mathcal{A}_T(X)$ such as Borel localization, a Chang–Skjelbred lemma, and a Goresky–Kottwitz–MacPherson presentation. In the special case of Hamiltonian torus actions on symplectic manifolds we prove a surjectivity criterion for the assignment equivariant Kirwan map corresponding to a circle in T. We then obtain a Tolman–Weitsman type presentation of the kernel of this map.

Keywords: Continuous group actions; Polynomial assignments; Equivariant cohomology; Symplectic quotients; Kirwan map

1. Introduction

The notion of assignment associated to a torus action on a manifold was defined by Ginzburg, Guillemin, and Karshon in [8], by means of a construction that takes into account exclusively the orbit stratification and the relative position of the strata. They were led to this construction while dealing with the existence problem of an abstract moment map for a given action. However, as

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http://dx.doi.org/10.1016/j.indag.2017.09.005

the authors briefly mention, this new notion is susceptible to be relevant for another important question in this area, namely, under which circumstances is the equivariant cohomology algebra determined by the orbit stratification? Indeed, a few years later, Guillemin, Sabatini, and Zara have found in [14] a direct connection between the equivariant cohomology and a particular assignment space, which is called by them the algebra of *polynomial assignments*. Concretely, the connection is given by a ring homomorphism, which, for certain classes of actions, is injective and sometimes even bijective. For example, injectivity is achieved for equivariantly formal actions with isolated fixed points on compact manifolds and bijectivity for the sub-class of actions of Goresky–Kottwitz–MacPherson (GKM) type.

This paper is based on the observation that the polynomial assignment algebra can be defined for arbitrary continuous (torus) actions on topological spaces. More precisely, let X be a topological space and T a torus that acts on it. For any $p \in X$ we denote by T_p the corresponding isotropy subgroup of T and by $\mathfrak{t}_p$ its Lie algebra (this will be referred to as the infinitesimal isotropy at p). Let also $S(\mathfrak{t}_p^*)$ be the algebra of polynomial functions on $\mathfrak{t}_p$.

Definition 1.1. A (polynomial) assignment for the T-action on X is a map A that assigns to each $p \in X$ a polynomial $A(p) \in S(\mathfrak{t}_p^*)$ such that for any (connected) subtorus $H \subset T$ the map $A^{\mathfrak{h}}$ on the fixed point set X^H is locally constant. Here $\mathfrak{h}$ is the Lie algebra of H and $A^{\mathfrak{h}}$ the map defined by $A^{\mathfrak{h}}(p) := A(p)|_{\mathfrak{h}}$, for all $p \in X^H$.

This looks different from the definition in [14] since, as already mentioned, the latter involves the orbit stratification. However, we will show in Section 3 that for smooth actions on manifolds, there is no difference between the two notions.

We denote by $\mathcal{A}_T(X)$ the space of all assignments of the above type. It has an obvious canonical structure of an $S(\mathfrak{t}^*)$-algebra, which will be referred to as the *assignment algebra* of the torus action. It defines a functor from the category of topological T-spaces to the category of $S(\mathfrak{t}^*)$-algebras; moreover, it is an equivariant homotopy invariant, see Section 2. Our goal here is to present some results concerning $\mathcal{A}_T(X)$ in the topological set-up. Direct connections with the equivariant cohomology algebra $H_T^*(X)$ in the spirit of [14] are also discussed, although they are not of main interest for us. Polynomial assignments are studied here in their own right.

In fact, equivariant cohomology is rather relevant for us in an indirect way: that is, we consider some results in this theory and prove assignment versions of them. In the first part we will consider the inclusion of the fixed point set X^T into X along with the map $\mathcal{A}_T(X) \to \mathcal{A}_T(X^T)$ induced by functoriality. After proving Borel type localization results, concerning the kernel and the cokernel of the aforementioned map, we obtain an assignment version of the GKM-theorem. It requires some extra assumptions on the (continuous) torus action. Among others, we want the fixed point set X^T to have only finitely many components, call them $Z_1, \dots, Z_n$. Then the theorem says that $\mathcal{A}_T(X)$ is isomorphic to the subspace of $S(\mathfrak{t}^*) \times \cdots \times S(\mathfrak{t}^*)$ (n factors) consisting of tuples $(f_1, \dots, f_n)$ with the property that if Z_i and Z_j are contained in a connected component of some X^H, where $H \subset T$ is a codimension one subtorus of Lie algebra $\mathfrak{h}$, then f_i and f_j are equal when restricted to $\mathfrak{h}$. The precise statement can be found in Section 5.3. We emphasize that the result is purely topological. One class of torus actions for which it holds true is the one of equivariantly formal actions on compact Hausdorff spaces with finitely many infinitesimal isotropies and finite dimensional cohomology. We note that this is in the spirit of [8, Section 3.4]: a characterization of $\mathcal{A}_T(X)$ similar to the one above is obtained there under the (more restrictive) hypotheses that X is a manifold, the T-action is smooth, and the compatibility relations are assumed for subtori H of arbitrary dimension.

Please cite this article in press as: O. Goertsches, A.-L. Mare, Assignments for topological group actions, Indagationes Mathematicae (2017), http://dx.doi.org/10.1016/j.indag.2017.09.005.