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Five ways to solve the Yoshida jet problem of wind-driven equatorial flow

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ABSTRACT

The Yoshida jet is a prototype for wind-driven flow in the tropical ocean. We review the Hermite solution discovered sixty years ago, and show how series acceleration through the Hutton-Moore or Euler schemes is necessary to obtain useful accuracy. An explicit solution in terms of Bessel functions is given here for the first time. It is also shown that much simpler analytic solutions, approximate but accurate, are given by low order rational Chebyshev series and Two-Point Padé approximants. Numerically, the Fourier sine domain truncation method combined with the change of coordinate $y \in \sinh(L\ t)$ gives quasi-geometric convergence for most problems. When applied to the Yoshida jet, however, the sine/sinh rate of convergence is awful compared to that of the rational Chebyshev spectral method because wind-driven tropical flows decay very slowly as $O(1/y)$. The log-weakened-geometric rate of convergence of the sine-sinh method, realized for functions that decay exponentially fast for large $|y|$, is replaced by root-exponential convergence with total error proportional to $\exp(-q\sqrt{N})$ for some positive constant q. Except for the explicit Bessel solutions, similar considerations should apply to tropical wind-driven flows in general.

1. Introduction

The earliest work on equatorially trapped dynamical phenomenon is the now-classic 1959 paper of Kozo Yoshida (1922–1978), who showed that the wind could drive a strong equatorial jet (Yoshida, 1959, 1962, 1967). The Equatorial Undercurrent, then called the Cromwell Current, was a recent discovery: a strong, shallow jet with velocities as high as 1.5 m/s—extraordinarily high for the sea—but with a very narrow latitudinal width of only a couple of hundred kilometers—and centered right at the equator.

The oceanographic significance of his work is well-discussed in the review article by Moore and Philander (1977) and in Chapter 9 of the author's book (Boyd, 2018). The methodological significance of Yoshida's work is that he was the first to show that problems in the idealized theory of tropical dynamics in the ocean and atmosphere could be solved almost trivially through Hermite series.

In this article, we show that ironically Yoshida's infinite Hermite series is unnecessary because his idealized problem can be solved explicitly in terms of Bessel functions. Furthermore, his Hermite series are very slowly convergent and are indeed almost useless without what are variously called “series acceleration”, “summability” or “sequence acceleration” methods. Boyd and Moore described the application of these to Hermite series thirty years ago; we show that better new accelerations improve their methods.

Indeed, Yoshida's problem can be attacked from many directions. We show that rational Chebyshev spectral methods and two-point Padé approximations both are surprisingly accurate at low order.

Yoshida's problem can be reduced to the ordinary differential equation for the north-south velocity.

$$v_{yy} - y^2 v = y, \qquad y \in [-\infty, \infty], \quad |v(y)| \to 0 \text{ as } |y| \to \infty \tag{1}$$

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Our focus will be almost entirely upon this ordinary differential equation, but the full solution to the time-dependent Yoshida problem and its oceanic context is given in the appendix. In the next section, we briefly review the problem and its "traditional" solution by infinite Hermite series. In the following two sections, we describe two modifications that can recover an exponential rate of convergence: sum acceleration by the Euler or other filters.

In the following sections, we discuss the Bessel function solutions, solutions using a basis of rational Chebyshev functions, and power series and asymptotic expansions converted into two-point Padé approximations.

2. Bessel function explicit solution for the Yoshida jet

The differential equation

$$v_{yy} - y^2 v = f(y) \tag{2}$$

can be transformed by the change of coordinate $\tilde{y} = (1/2)y^2$ into a differential equation whose homogeneous solutions are the $\sqrt{\tilde{y}}$ times a modified Bessel function of $\tilde{y}$. (Identity 9.1.52 on pg. 362 of Abramowitz and Stegun (1965).) The transformed equation has the general solution

$$v = C_I \sqrt{y}\, \mathrm{I}_{1/4}((1/2)y^2) + C_K \sqrt{y}\, \mathrm{K}_{1/4}((1/2)y^2) + v^p(y) \tag{3}$$

$$\begin{aligned} v^p = & \ \tfrac{1}{2}\sqrt{y}\left\{\tfrac{1}{2}\sqrt{y}\,(\mathrm{I}_{1/4}((1/2)y^2)\int \sqrt{z}\,\mathrm{K}_{1/4}((1/2)z^2)f(z)\mathrm{dz} \right. \\ & \left. - \mathrm{K}_{1/4}((1/2)y^2)\int \sqrt{z}\,\mathrm{I}_{1/4}((1/2)z^2)f(z)\mathrm{dz})\right\} \end{aligned} \tag{4}$$

Because the K-Bessel function is singular at the origin, $C_K = 0$. The remaining constant C_I is chosen to enforce the boundary condition $v(y) \to 0$ as $|y| \to \infty$, exploiting the known asymptotic approximations of the Bessel functions.

For the particular case of $f(y) = y$,

$$v = C_I \sqrt{y}\, \mathrm{I}_{1/4}([1/2]y^2) + v^p(y) \tag{5}$$

$$C_I = -1/4\sqrt{2\pi}\ \ \Gamma(3/4) = -.767916038651074 \tag{6}$$

$$v^p = 1/8\sqrt{2\pi}\,\Gamma(3/4)y^{5/2}\mathbf{L}_{1/4}([1/2]y^2)\times \tag{7}$$

$$(\mathrm{K}_{3/4}(1/2]y^2)\mathrm{I}_{1/4}([1/2]y^2) + \mathrm{K}_{1/4}(1/2y^2)\mathrm{I}_{-3/4}([1/2]y^2)) \tag{8}$$

where $\mathbf{L}_{1/4}(z)$ is the usual modified Struve function which solves $z^2u_{zz} + zu_z - (z^2 + (1/16))u = \{2^{3/4}/(\sqrt{\pi}\,\Gamma(3/4))\}z^{5/4}$.

Although these solutions appear complicated, routines to evaluate Bessel functions are included in Matlab, Maple and all Fortran and C+ + libraries. The Struve function has power series and asymptotic approximations and the integral representation

$$\mathbf{L}_{1/4}(z) = \frac{2^{3/4}}{\Gamma(3/4)\sqrt{\pi}} z^{1/4} \int_0^{\pi/2} \mathrm{d}t \sinh(z\cos(t))\sqrt{\sin(t)} \tag{9}$$

The function v found here is identical with v_{sl}, the part of the north-south velocity of the Yoshida flow which is independent of time

$$v_{\mathrm{sl}} = v \tag{10}$$

$$u_{\mathrm{sl}} = t(1 + yv_{\mathrm{sl}}) \tag{11}$$

$$\phi_{\mathrm{sl}} = t(-v_{y,\mathrm{sl}}) \tag{12}$$

$$\begin{aligned} \phi_{\mathrm{sl}}/t \ = \ & -1/24y^{3/2}(-6\sqrt{2}\sqrt{\pi}\Gamma(3/4)\mathrm{I}_{-3/4}(1/2y^2) \\ & + 3\sqrt{2}\sqrt{\pi}\Gamma(3/4)\mathbf{L}(1/4,\ 1/2y^2)\mathrm{K}_{3/4}(1/2y^2)\mathrm{I}_{1/4}(1/2y^2) \\ & + 3\sqrt{2}\sqrt{\pi}\Gamma(3/4)\mathbf{L}(1/4,\ 1/2y^2)\mathrm{K}_{1/4}(1/2y^2)\mathrm{I}_{-3/4}(1/2y^2) \\ & + 3\sqrt{2}\sqrt{\pi}\Gamma(3/4)\mathbf{L}(5/4,\ 1/2y^2)y^2\mathrm{K}_{3/4}(1/2y^2)\mathrm{I}_{1/4}(1/2y^2) \\ & + 3\sqrt{2}\sqrt{\pi}\Gamma(3/4)\mathbf{L}(5/4,\ 1/2y^2)y^2\mathrm{K}_{1/4},\ (1/2y^2)\mathrm{I}_{-3/4}(1/2y^2) \\ & + 4y^2\sqrt[4]{y^2}\,\mathrm{K}_{3/4}(1/2y^2)\mathrm{I}_{1/4}(1/2y^2) \\ & + 4y^2\sqrt[4]{y^2}\,\mathrm{K}_{1/4}(,1/2y^2)\mathrm{I}_{-3/4}(1/2y^2)) \end{aligned} \tag{13}$$

3. Forced eigenoperators: Hermite series

It is easy to solve Yoshida's problem with a basis of orthonormal Hermite functions because the differential operator is the Hermite eigenoperator:

$$\left\{\frac{d^2}{dy^2} - y^2\right\}\psi_n(y) = -(2n+1)\psi_n(y) \tag{14}$$