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Spot volatility estimation using the Laplace transform

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ABSTRACT

A new non-parametric estimator of the instantaneous volatility is defined relying on the link between the Laplace transform of the price process and that of the volatility process for Brownian semimartingale models. The proposed estimation method is a global one, in the spirit of methods based on Fourier series decomposition, with a *plus* for improving the precision of the volatility estimates near the boundary of the time interval. Consistency and asymptotic normality of the proposed estimator are proved. A simulation study confirms the theoretical results and Monte Carlo evidence of the favorable performance of the proposed estimator in the presence of microstructure noise effects is presented.

1. Introduction

The Laplace transform has been widely used in the literature on option pricing (see, e.g., Carr and Wu, 2004; Fusai, 2004; Leblanc and Scaillet, 1998; Lee, 2004). More recently, Tauchen and Todorov (2012) introduced the empirical Laplace transform to study the characteristics of the volatility. Nevertheless, the Laplace transform methodology has not been used to estimate the instantaneous volatility. As shown here, a suitable convolution product of the Laplace transform of asset returns is an appropriate tool to build global non-parametric estimators of the instantaneous volatility.

The first steps in the harmonic analysis method to compute instantaneous multivariate volatilities were made by Malliavin and Mancino (2002). A peculiarity of the Fourier estimation procedure is that, in virtue of its own definition, it uses all available observations and avoids any manipulation of the original data (as happens with several alternative proposals which rely on data-synchronization methods), because it is based on the integration rather than differentiation of the time series of prices. Moreover, the Fourier estimation method is a *global method*, unlike methods based on the use of observations in local time windows. This approach allows us to overcome the use of the empirical derivative employed in a vast range of literature to obtain the instantaneous volatility (see the recent book by Ait-Sahalia and Jacod, 2014). A consistent estimator of the spot volatility was defined by Malliavin and Mancino (2009) relying on the finite Fourier transform of the price process over a fixed time interval in the absence of microstructure noise effects. The efficiency of the Fourier method

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in the presence of microstructure noise has been analyzed and compared with other estimators by Nielsen and Frederiksen (2008) and Mancino and Sanfelici (2008; 2011). The authors show that the Fourier estimator of integrated co-volatilities is substantially unaffected by the presence of microstructure noise contaminations by suitably choosing the number of Fourier coefficients to include in the estimator. More recently, Park et al. (2016) introduce the *Fourier Realized Kernel* estimator and show that it is consistent even in the presence of microstructure noise effects. Limit theorems for the Fourier estimator of integrated multivariate volatility under different sampling schemes are proved by Clement and Gloter (2011), whereas Cuchiero and Teichmann (2013) extend the Fourier method in the presence of jumps.

The Laplace transform method hinges on the same two-step procedure used for the Fourier estimation approach, that is, the convolution product of an integral transform of the asset returns and an inversion formula. Therefore, it presents the same advantage as the Fourier estimation approach with respect to the quadratic variation-based methods, concerning the use of all the available observations without the need for any manipulation of the original data. It also has additional attractive features. From a conceptual point of view, the introduction of the Laplace transform produces two benefits. Firstly, it avoids the artificial "periodization" of the asset price process subjacent to Fourier series methodology, which is responsible for the low precision of the estimate near the boundary of the time interval. Secondly, it leads to the estimator defined in (8), which constitutes a bridge between the two different approaches to compute the volatility, namely, *local* methods based on the quadratic variation formula and the *global* approach via Fourier analysis.

The main analytical result proved here is that under the hypothesis that the price process is a continuous semi-martingale, the Laplace transform of the stochastic volatility function is equal to the Bohr convolution product of the Laplace transform of the price process. Consequently, the transform needs to be inverted to obtain the spot volatility estimator. As a matter of fact, computing the Laplace transform of a given function corresponds to computing the Fourier transform of the function multiplied by a damping exponential factor. Therefore, the Fourier inversion formula is applied to obtain the (damped) volatility function. The proposed estimator considers a long time series of prices by smoothing past data. This procedure generates a spot volatility estimator that employs weighted sums of squared and cross increments of the price process. The quadratic term is like the triangular kernel-based realized estimator studied by Fan and Wang (2008) and Kristensen (2010); however, the convolution product also generates off-diagonal cross products.

We prove that the Laplace estimator of spot volatility is statistically efficient in terms of rate of convergence and asymptotic variance. Under a suitable choice of the relative growth between the number of data, the convolution frequency and the bandwidth, the Laplace estimator has asymptotic variance equal to $(4/3)\sigma^4(t)$ (where $\sigma(t)$ denotes the volatility process) which is the same for the triangular kernel-based realized estimator and the same rate of convergence (see the result by Fan and Wang, 2008). Therefore, we prove that the effect of the cross terms is not detrimental in view of the asymptotic efficiency, if the couple convolution frequency and bandwidth are appropriately selected as indicated in this theory. On the other hand, it is known that the role of the cross terms is crucial for the estimator's robustness in the presence of microstructure noise in the case of the Fourier estimator in Mancino and Sanfelici (2008), the Fourier realized kernel estimator in Park et al. (2016), and the realized kernels in Barndorff-Nielsen et al. (2008). As it concerns the Laplace method, Monte Carlo evidence of their contribution is provided by showing that the Laplace estimator outperforms the triangular kernel-based realized estimator in the presence of microstructure noise both inside the interval of observations and near the boundary.

The finite sample properties of the Laplace estimator are analyzed through an intensive simulation study. The frequency (which appears in the convolution) and the bandwidth (arising in the kernel) must be suitably selected to efficiently implement the estimator. Therefore, a method is proposed to select them in a feasible way by using the realized Laplace transform of volatility introduced by Tauchen and Todorov (2012); numerical evidence that the feasible estimator shows the same performance as the non-feasible one is provided.

The paper is organized as follows. Section 2 contains the main result: the Laplace transform of the volatility function is computed to be the Bohr convolution product of the Laplace transform of the log-price. Given discrete unevenly spaced observations of the price, Section 3 provides the explicit expression of the Laplace estimator with proofs for consistency and asymptotic normality. Section 4 contains the simulation study and Section 5 the conclusions. The proofs are in Appendix A, whereas Appendix B contains some auxiliary lemmas.

The main idea presented here originated during discussions the second author had with Prof. Paul Malliavin when he visited the Scuola Normale Superiore of Pisa in 2004 and it was outlined by Malliavin et al. (2005). We recently decided to further explore this promising idea and, not surprisingly, we found it very interesting. We thus dedicate this work to the memory of Prof. Paul Malliavin with our gratitude.

2. The Laplace transform of volatility

In this section, given a continuous trajectory of the asset price process (continuous semi-martingale model), the Laplace transform of the (latent) volatility process is computed. In fact, this analytical result is key to constructing the spot volatility estimator in the next section.

The evolution of the logarithm asset price process $p(t)$ is described by the stochastic differential equation

$$dp(t) = \sigma(t)dW(t) + b(t)dt, \tag{1}$$

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