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Higher-order statistics for DSGE models☆

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ABSTRACT

Closed-form expressions for unconditional moments, cumulants and polyspectra of order higher than two are derived for non-Gaussian or nonlinear (pruned) solutions to DSGE models. Apart from the existence of moments and white noise property no distributional assumptions are needed. The accuracy and utility of the formulas for computing skewness and kurtosis are demonstrated by three prominent models: the baseline medium-sized New Keynesian model used for empirical analysis (first-order approximation), a small-scale business cycle model (second-order approximation) and the neoclassical growth model (third-order approximation). Both the Gaussian as well as Student's *t*-distribution are considered as the underlying stochastic processes. Lastly, the efficiency gain of including higher-order statistics is demonstrated by the estimation of a RBC model within a Generalized Method of Moments framework.

1. Introduction

Most macroeconomic time series do not follow the Gaussian distribution but are rather characterized by asymmetry and thick tails. For instance, consumption price indices and interest rates can typically be described by skewed distributions, whereas consumption exhibits excess kurtosis compared to a normal distribution. Furthermore, growth rates are seldom Gaussian, a point emphasized by Fagiolo et al. (2008). Current workhorse DSGE models are, however, linearized and one assumes the normal distribution for the underlying stochastic innovations and structural shocks (e.g. Smets and Wouters, 2007). This typical approach is attractive, since the resulting state space representation is a linear Gaussian system. Using the Kalman filter one can then use either Maximum Likelihood (see e.g. Andreasen, 2010) or Bayesian (see e.g. An and Schorfheide, 2007) methods to efficiently estimate these models in a full-information estimation strategy. In a limited-information estimation strategy (General Method of Moments (GMM), Simulated Method of Moments (SMM) or Indirect Inference, see e.g. Ruge-Murcia, 2007) estimation is focused on the first two moments of data, since a Gaussian process is completely characterized by its mean and (co-)variance. This, however, cannot capture important features of macroeconomic time series behavior. Ascari et al. (2015) show that simulated data from standard linearized DSGE models with either Gaussian or Laplace distributed shocks fail to replicate asymmetry and thick tails one observes in real data. Accordingly, Christiano (2007) finds strong evidence against the normality assumption based on the skewness and kurtosis properties of residuals in an estimated VAR model. Implications of models that are not able to depict asymmetry and heavy tails in

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their data-generating-process are hence not reliable and should be used only with care for policy evaluation. DSGE models should therefore not only replicate the first two moments of data, but also higher-order statistics such as skewness and kurtosis.

Basically, there are two complementary approaches to overcome this shortcoming. For one, we can discard the Gaussianity assumption. Accordingly, Chib and Ramamurthy (2014) and Curdia et al. (2014) estimate standard linear DSGE models with Student's *t*-distributed shocks and conclude that these models outperform their Gaussian counterparts. On the other hand, we can relax the linearity assumption and use a nonlinear solution to the DSGE model. In both cases it is natural to analyze whether we are able to exploit information from higher-order moments for the calibration, estimation and identification of parameters. Researchers in mathematics, statistics and signal processing have developed tools, called higher-order statistics (HOS), to solve detection, estimation and identification problems when the noise source is non-Gaussian or we are faced with nonlinearities; however, applications in the macroeconometric literature are rather sparse. Introductory literature and tutorials on HOS can be found in the textbooks of Brillinger (2001), Nikias and Petropulu (1993), Priestley (1983) and the references therein. The basic tools of HOS are cumulants, which are defined as the coefficients in the Taylor expansion of the log characteristic function in the time-domain; and polyspectra, which are defined as Fourier transformations of the cumulants in the frequency-domain. Cumulants and polyspectra are attractive for several reasons. For instance, all cumulants and polyspectra of a Gaussian process of order three and above are zero, whereas the same applies only to odd product-moments. Furthermore, the cumulant of two statistically independent random processes equals the sum of the cumulants of the individual processes (which is not true for higher-order moments). And lastly, cumulants of a white noise sequence are Kronecker delta functions, so that their polyspectra are flat (Mendel, 1991). For a mathematical discussion of using cumulants instead of moments in terms of ergodicity and proper functions, see Brillinger (1965). Note that if two probability distributions have the same moments, they will have the same cumulants as well.

In this paper, we derive closed-form expressions for unconditional third- and fourth-order moments, cumulants and corresponding polyspectra for non-Gaussian or nonlinear DSGE models. We limit ourselves to fourth-order statistics, since third-order cumulants and the bispectrum capture nonlinearities (or non-Gaussianity) for a skewed process, whereas the fourth-order cumulants and the trispectrum can be used in the case of a non-Gaussian symmetric probability distribution. Regarding the approximation of the nonlinear solution to DSGE models we focus on the pruning scheme proposed by Kim et al. (2008) and operationalized by Andreasen et al. (2016), since the pruned state space (PSS from now on) is a linear, stationary and ergodic state space system. In the PSS, however, Gaussian innovations do not imply Gaussian likelihood, leaving scope for higher-order statistics to capture information from nonlinearities and non-Gaussianity.

This paper is not the first to provide closed-form expressions for unconditional moments in higher-order approximated and pruned solutions to DSGE models. Schmitt-Grohé and Uribe (2004) implicitly use pruning in their code to compute unconditional first two moments for a second-order approximation. Likewise Lan and Meyer-Gohde (2013a) provide methods to compute unconditional first two moments based on Volterra series expansions. Closest to our approach (and which we take as a starting point) is Andreasen et al. (2016). They show how to set up the PSS for any order of approximation and provide closed-form expressions and code to compute unconditional first two moments in the PSS. These three algorithms, however, rely on the Gaussian distribution as the underlying shock process (not necessarily conceptually but at least in the corresponding algorithms), whereas our symbolic script files can be used for any distribution provided the relevant moments exist. We extensively tested our procedures with the ones in Andreasen et al. (2016) and found that when using the Gaussian distribution and the same algorithm for Lyapunov equations the first two moments are identical. Our paper is, however, the first to provide closed-form expressions and code for the computation of unconditional moments higher than two as well as corresponding cumulants and polyspectra.

Accordingly, we demonstrate our procedures by means of the Smets and Wouters (2007) model for a first-order approximation, the An and Schorfheide (2007) model for a second-order approximation and the canonical neoclassical growth model, e.g. Schmitt-Grohé and Uribe (2004), for a third-order approximation. For all models we consider both the Gaussian as well as Student's *t*-distribution with thick tails as the underlying shock process and compare our theoretical results with simulated higher-order moments. We focus particularly on skewness and excess kurtosis in our simulations, since these are typical measures an applied researcher would like to match in a calibration exercise. On the other hand auto- and cross-(co-)skewness as well as kurtosis may contain valuable information in an estimation exercise, see e.g. Harvey and Siddique (2000). Therefore, we illustrate our analytical expressions for higher-order statistics within a GMM estimation exercise. We demonstrate the efficiency gain of including third-order product moments in the estimation of a Real Business Cycle (RBC) model with habit formation and variable labor.

The paper is structured as follows. Section 2 sets up the general DSGE framework and discusses linear as well as nonlinear solution methods. The derivations of the PSS are given in Section 3. A univariate example is used make the procedure of pruning illustrative. In Section 4, we provide formal definitions and establish notation regarding univariate and multivariate cumulants and polyspectra. In this manner, we are able to derive closed-form expressions for unconditional moments, cumulants and polyspectra up to order four for linear and nonlinear (pruned) solutions to DSGE models in Section 5. The accuracy and utility of the formulas for computing skewness and kurtosis are demonstrated in Section 6. In the following Section 7, we illustrate the efficiency gain of including higher-order statistics within a GMM estimation. Section 8 concludes and points out interesting applications for the proposed algorithm and results. Our DYNARE toolbox for the computation of higher-order statistics and for the GMM estimation is model-independent and can be used for DSGE models solved up to a third-order approximation.