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Connected graph G with $\sigma_2(G) \geq \frac{2}{3}n$ and $K_{1,4}$-free contains a Hamiltonian path

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ABSTRACT

Let $G = (V_G, E_G)$ be a connected graph with n vertices. We show that if $\sigma_2(G) \geq \frac{2}{3}n$ and G is $K_{1,4}$-free (i.e. without induced $K_{1,4}$), then G contains a Hamiltonian path.

1. Introduction

The Hamiltonian problem; determining conditions under which a graph contains a spanning path or cycle has long been fundamental in graph theory. Named for Sir William Rowan Hamilton, this problem traces its origins to the 1850s. Today, however, the constant stream of results in this area continues to supply us with new and interesting theorems and still further questions. Since this area is so vast, I shall certainly be unable to mention everything and I shall expect my reader to be somewhat familiar with this area.

There are three fundamental results that I feel deserve special attention here; both for their contribution to the overall theory and for their affect on the development of the area. In many ways, these three results are the foundation of much of today's work.

Theorem 1 (*Dirac - 1952 - [4]*)**.** *A graph on $n \geq 3$ vertices in which each vertex has degree at least $\frac{n}{2}$ contains a Hamiltonian cycle.*

Theorem 2 (*Ore - 1960 - [10]*)**.** *A graph on $n \geq 3$ vertices in which for every pair of nonadjacent vertices, the sum of their degrees is at least n, contains a Hamiltonian cycle.*

Theorem 3 (*Bondy–Chvátal - 1976 - [1]*)**.** *A graph G on n vertices contains a Hamiltonian cycle if and only if the graph uniquely constructed from G by repeatedly adding a new edge connecting a pair of nonadjacent vertices with sum of their degrees at least n until no more pairs with this property can be found, contains a Hamiltonian cycle.*

These original results started a new approach to develop sufficient conditions on degrees for a graph to have Hamiltonian path or cycle. A lot of effort have been made by various people in generalization of these theorems and this area is one of the core subjects in Hamiltonian graph theory and extremal graph theory. For more results in this area, see [5–7] or [8].

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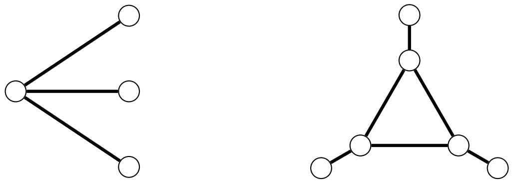
It is natural to ask if strengthening the connectivity conditions would allow us to lower the degree conditions. I will concentrate my efforts on results and problems dealing with spanning paths in connected graphs. I shall not attempt to survey paths in connected graphs. However, I can mention this related result (see [6]).

Theorem 4 (*Duffus–Jacobson–Gould- 1982*)**.** *A connected,* $\{K_{1,3}, N\}$*-free graph contains a Hamiltonian path.*



$K_{1,3}$ N

In this paper, we present a sufficient condition for a connected graph to have a Hamiltonian path.

2. Preliminary definitions

We refer to [2] or [3] for undefined notations. The graphs $G = (V_G, E_G)$ considered in this paper are undirected, simple. The *size* of a graph is its number of vertices. For a graph $G = (V_G, E_G)$ and $u, v \in V_G$ we define

$$G + uv = (V_G, E_G \cup \{uv\}) \text{ and } G - uv = (V_G, E_G \setminus \{uv\}).$$

A *path* $P = (V_P, E_P)$ in G is a nonempty subgraph P of the form

$$V_P = \{v_1, \dots, v_k\} \subset V_G \text{ and } E_P = \{v_i v_{i+1} : i \in \{1, \dots, k-1\}\} \subset E_G.$$

A *cycle* C in G is a subgraph of the form $P + v_1 v_k$ where P is a path. We often use the notations $P = v_1, v_2, \dots, v_k$ and $C = v_1, v_2, \dots, v_k, v_1$ The *neighbourhood* of a vertex v in a graph G is $N_G(v) = \{u : uv \in E_G\}$. The degree of v is $deg_G(v) = |N_G(v)|$. A *dominating cycle* C in G is such that every vertex of the graph has at least one adjacent vertex on the cycle *i.e.* $\bigcup_{v \in V_C} N_G(v) = V_G$. An *(vertex-)induced subgraph* of G is a subset of V_G together with any edges whose endpoints are both in this subset. For a graph F, we say that G *contains* F if G contains an induced subgraph isomorphic to F and G is F*-free* if it does not contain F.

For $1 \leq k \leq |V_G|$ we define

$$\sigma_k(G) = \min\{\sum_{i=1}^{k} deg_G(v_i) : v_1, \dots, v_k \text{ are pairwisely nonadjacent vertices}\},$$

with the convention $\min\{\emptyset\} = +\infty$.

3. Our results

Lemma 1. *Let G be a graph on n vertices and $1 \leq k \leq k' \leq n$. We have*

$$\sigma_{k'}(G) \geq \frac{k'}{k}\sigma_k(G).$$

Proof. We will show that for $1 \leq k \leq n-1$,

$$\sigma_{k+1}(G) \geq \frac{k+1}{k}\sigma_k(G)$$

and the result will follow recursively. If $\sigma_{k+1}(G) = +\infty$ the result is true. Else, let $v_1, \dots, v_{k+1}$ be pairwisely nonadjacent vertices of G. We have

$$k\sum_{i=1}^{k+1} deg_G(v_i) = \sum_{\substack{I \subset \{1,\dots,k+1\} \\ |I|=k}} \sum_{i \in I} deg_G(v_i) \geq \binom{k+1}{k}\sigma_k(G) = (k+1)\sigma_k(G),$$

and by dividing by k

$$\sum_{i=1}^{k+1} deg_G(v_i) = \frac{k+1}{k}\sigma_k(G).$$