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Cactus graphs with minimum edge revised Szeged index[☆]

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ABSTRACT

The edge revised Szeged index $Sz_e^*(G)$ is defined as $Sz_e^*(G) = \sum_{e=uv\in E}(m_u(e) + m_0(e)/2)(m_v(e)+m_0(e)/2)$, where $m_u(e)$ and $m_v(e)$ are, respectively, the number of edges of G lying closer to vertex u than to vertex v and the number of edges of G lying closer to vertex v than to vertex u, and $m_0(e)$ is the number of edges equidistant to u and v. A cactus graph is a connected graph in which every block is an edge or a cycle. In this paper, we give a lower bound of the edge revised Szeged index among all m-edges cactus graphs with k cycles, and also characterize those graphs that achieve the lower bound. We also obtain the second minimum edge revised Szeged index for connected cactus graphs of size m with k cycles.

1. Introduction

All graphs considered in this paper are finite, undirected and simple. We refer the readers to [2] for terminology and notations. Let C_n denote the cycle on n vertices. Let G be a connected graph with vertex set V and edge set E. Call u a *pendant vertex* of G, if $d_G(u) = 1$ and call uv a *pendant edge* of G, if $d_G(u) = 1$ or $d_G(v) = 1$. For $u, v \in V$, $d(u, v)$ denotes the distance between u and v. The *Wiener index* of G is defined as

$$W(G) = \sum_{\{u,v\}\subseteq V} d(u, v).$$

This topological index has been extensively studied in the mathematical literature; see, e.g., [8,10]. Let $e = uv$ be an edge of G, and define three sets as follows:

$$N_u(e) = \{w \in V : d(u, w) < d(v, w)\},$$

$$N_v(e) = \{w \in V : d(v, w) < d(u, w)\},$$

$$N_0(e) = \{w \in V : d(u, w) = d(v, w)\}.$$

Thus, $\{N_u(e), N_v(e), N_0(e)\}$ is a partition of the vertices of G with respect to e. The numbers of vertices of $N_u(e)$, $N_v(e)$ and $N_0(e)$ are denoted by $n_u(e)$, $n_v(e)$ and $n_0(e)$, respectively. Evidently, if n is the number of vertices of the graph G, then $n_u(e) + n_v(e) + n_0(e) = n$. A long time known property of the Wiener index is the formula [9,21]:

$$W(T) = \sum_{e=uv\in E} n_u(e)n_v(e),$$

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which is applicable for trees T. Using the above formula, Gutman [6] introduced a graph invariant named the *Szeged index* as an extension of the Wiener index and defined it by

$$Sz(G)=\sum_{e=uv\in E} n_u(e)n_v(e).$$

Randić [16] observed that the Szeged index does not take into account the contributions of the vertices at equal distances from the endpoints of an edge, and so he conceived a modified version of the Szeged index which is named the *revised Szeged index*. The revised Szeged index of a connected graph G is defined as

$$Sz^*(G)=\sum_{e=uv\in E}\left(n_u(e)+\frac{n_0(e)}{2}\right)\left(n_v(e)+\frac{n_0(e)}{2}\right).$$

Some properties and applications of these topological indices have been reported in [1,4,12,14,15,17,19,20,22].

Given an edge $e=uv\in E$, the distance between the edge e and the vertex x, denoted by $d(e,x)$, is defined as

$$d(e,x)=\min\{d(u,x),d(v,x)\}.$$

Similarly, the sets $M_0(e)$, $M_u(e)$ and $M_v(e)$ are defined to be the set of edges equidistant from u and v, the set of edges whose distance to vertex u is smaller than the distance to vertex v and the set of edges closer to v than u, respectively. The numbers of edges of $M_u(e)$, $M_v(e)$ and $M_0(e)$ are denoted by $m_u(e)$, $m_v(e)$ and $m_0(e)$, respectively. Evidently, if m is the number of edges of the graph G, then $m_u(e)+m_v(e)+m_0(e)=m$. The *edge Szeged index* [7], and the *edge revised Szeged index* [5] of G are defined as follows:

$$Sz_e(G)=\sum_{e=uv\in E} m_u(e)m_v(e),$$

$$Sz_e^*(G)=\sum_{e=uv\in E}\left(m_u(e)+\frac{m_0(e)}{2}\right)\left(m_v(e)+\frac{m_0(e)}{2}\right).$$

Results on edge Szeged index can be found in [3,11,18]. In [5], Dong et al. determined the n-vertex unicyclic graphs with the largest and the smallest revised edge Szeged indices. In [13], Liu and Chen gave an upper bound of the edge revised Szeged index for a connected bicyclic graphs, and also characterized those graphs that achieve the upper bound.

A cactus graph is a graph in which every block is an edge or a cycle. It is also a graph whose all cycles are edge-disjoint. A cycle in a cactus is called an *end-block* if all but one vertex of this cycle have degree 2. Let $\mathcal{C}(m,k)$ be the class of all cactus graphs of size m with k cycles. In this paper, we give a lower bound of the edge revised index in $\mathcal{C}(m,k)$, and characterize the extremal graphs that achieve the lower bound. We also obtain the second minimum edge revised Szeged index for connected cactus graphs of size m with k cycles.

2. Preliminaries

In this section, we give some preliminary results which will be used in the next sections to prove our theorems. We always denote the number of edges of G by m, i.e. $m=|E(G)|$. Using the fact that $m_u(e)+m_v(e)+m_0(e)=m$, we have

$$\begin{aligned}Sz_e^*(G)&=\sum_{e=uv\in E}\left(m_u(e)+\frac{m_0(e)}{2}\right)\left(m_v(e)+\frac{m_0(e)}{2}\right)\\&=\sum_{e=uv\in E}\left(\frac{m+m_u(e)-m_v(e)}{2}\right)\left(\frac{m-m_u(e)+m_v(e)}{2}\right)\\&=\sum_{e=uv\in E}\frac{m^2-(m_u(e)-m_v(e))^2}{4}\end{aligned}$$

So we get that

$$Sz_e^*(G)=\frac{m^3}{4}-\frac{1}{4}\sum_{e=uv\in E}(m_u(e)-m_v(e))^2 \tag{1}$$

Lemma 2.1. *Let G be a connected graph, and $e=uv$ be a cut edge of G. Then $(m_u(e)-m_v(e))^2\le(m-1)^2$ with equality if and only if $e=uv$ is a pendant edge.*

Proof. Let G_u and G_v be the components of $G-uv$ that contain u and v, respectively. It is obvious that $M_u(e)=E(G_u)$, $M_v(e)=E(G_v)$ and $M_0(e)=\{e\}$. Using the fact that $m_u(e)+m_v(e)+m_0(e)=m$, we have

$$(m_u(e)-m_v(e))^2=(|E(G_u)|-|E(G_v)|)^2=(m-1-2|E(G_v)|)^2\le(m-1)^2$$

with equality if and only if $|E(G_u)|=0$ or $|E(G_v)|=0$, that is, e is a pendant edge. ■