ARTICLE IN PRESS
Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

Rainbow spanning trees in properly coloured complete graphs

József Balogh [a,b], Hong Liu [c,*], Richard Montgomery [b]

[a] *Department of Mathematical Sciences, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA*
[b] *Trinity College, Cambridge, CB2 1TQ, UK*
[c] *Mathematics Institute and DIMAP, University of Warwick, Coventry, CV4 7AL, UK*

ARTICLE INFO

Article history:
Received 24 May 2017
Received in revised form 10 March 2018
Accepted 19 March 2018
Available online xxxx

Keywords:
Rainbow subgraph
Spanning trees

ABSTRACT

In this short note, we study pairwise edge-disjoint rainbow spanning trees in properly edge-coloured complete graphs, where a graph is rainbow if its edges have distinct colours. Brualdi and Hollingsworth conjectured that every K_n properly edge-coloured by $n-1$ colours has $n/2$ edge-disjoint rainbow spanning trees. Kaneko, Kano and Suzuki later suggested this should hold for every properly edge-coloured K_n. Improving the previous best known bound, we show that every properly edge-coloured K_n contains $\Omega(n)$ pairwise edge-disjoint rainbow spanning trees.

Independently, Pokrovskiy and Sudakov recently proved that every properly edge-coloured K_n contains $\Omega(n)$ isomorphic pairwise edge-disjoint rainbow spanning trees.

1. Introduction

Given a properly edge-coloured K_n, a *rainbow spanning tree* is a tree with vertex set $V(K_n)$ whose edges have distinct colours. Each properly edge-coloured K_n clearly contains many rainbow spanning trees—for example, any $(n-1)$-edge star in K_n is such a tree. How many *edge-disjoint* rainbow spanning trees can we find in any properly edge-coloured K_n? Brualdi and Hollingsworth [2] conjectured that every K_n properly edge-coloured by $n-1$ colours contains $n/2$ edge-disjoint rainbow spanning trees (see also Constantine [4]). Note that, in such a colouring, each colour appears on exactly $n/2$ edges to form a monochromatic perfect matching. Brualdi and Hollingsworth [2] showed that there are at least two edge-disjoint rainbow spanning trees in any K_n properly edge-coloured by $n-1$ colours.

Kaneko, Kano and Suzuki [6] expanded the Brualdi–Hollingsworth conjecture, suggesting that any properly edge-coloured K_n (using any number of colours) should contain $n/2$ edge-disjoint rainbow spanning trees. In such graphs, Kaneko, Kano and Suzuki [6] showed that there are at least three edge-disjoint rainbow spanning trees. Akbari and Alipour [1] studied edge-disjoint rainbow spanning trees using weaker conditions still, showing that any edge-coloured K_n, where each colour is only constrained to appear at most $n/2$ times, contains at least two edge-disjoint rainbow spanning trees.

Recent progress has seen far more edge-disjoint rainbow spanning trees found in edge-coloured complete graphs. For some small $\varepsilon > 0$, Horn [5] proved that every properly $(n-1)$-edge-coloured K_n contains at least εn edge-disjoint rainbow spanning trees. Carraher, Hartke and Horn [3] showed that every edge-coloured K_n where each colour appears at most $n/2$ times contains at least $\lfloor n/(1000 \log n) \rfloor$ edge-disjoint rainbow spanning trees. Here, we consider the intermediate of these conditions, and show that any properly edge-coloured K_n contains linearly many edge-disjoint rainbow spanning trees.

* Corresponding author.
E-mail addresses: jobal@math.uiuc.edu (J. Balogh), h.liu.9@warwick.ac.uk (H. Liu), r.h.montgomery@dpmms.cam.ac.uk (R. Montgomery).

https://doi.org/10.1016/j.dam.2018.03.066
0166-218X/

Theorem 1.1. *Every properly edge-coloured K_n contains at least $n/10^{12}$ pairwise edge-disjoint rainbow spanning trees.*

We note that our methods are much shorter than those previously capable of finding many edge-disjoint rainbow spanning trees. In particular, Section 4 alone shows that any properly $(n-1)$-edge-coloured K_n contains linearly many edge-disjoint rainbow spanning trees.

Essentially, to prove Theorem 1.1, we iteratively remove rainbow spanning trees from K_n while ensuring that the remaining minimum degree does not decrease too much and that we do not use up too many colours. We have two cases, depending on the colouring of K_n. We show (in Lemma 2.5) that every properly edge-coloured K_n either contains many colours which appear on linearly many edges or its colours can be grouped into classes which play a similar role. We use different embedding strategies for these cases (in Lemmas 2.6 and 2.7).

While finishing our work, we discovered that, using different methods, Pokrovskiy and Sudakov [7] recently proved that every properly edge-coloured K_n contains at least $n/10^6$ edge-disjoint rainbow copies of a certain spanning tree with radius 2.

Organisation: The rest of the paper is organised as follows. In Section 2, we present our main lemmas and show that they imply Theorem 1.1. These lemmas, Lemmas 2.5, 2.6 and 2.7, are proved in Sections 3, 4 and 5 respectively.

2. Preliminaries

Notation. For a graph G, denote by $|G|$ the number of vertices in G and by $\nu(G)$ the size of a largest matching in G. For each $A \subseteq V(G)$, let $G[A]$ and $G-A$ be the induced subgraph of G on the vertex set A and $V(G) \setminus A$ respectively. When F is a spanning subgraph of G, denote by $G \setminus F$ the spanning subgraph of G with edge set $E(G) \setminus E(F)$. For any set of colours U from an edge-coloured graph G, let G_U be the maximal subgraph of G whose edges have colour in U. We omit floor and ceiling signs when they are not essential.

The main tool we use to find rainbow spanning trees is the following result of Schrijver [8] and Suzuki [9].

Theorem 2.1. *Let G be an n-vertex edge-coloured graph. If, for every $1 \le s \le n$ and for every partition S of $V(G)$ into s parts, there are at least $s-1$ edges of different colours between the parts of S, then G contains a rainbow spanning tree.* □

Roughly speaking, we will iteratively find and remove edge-disjoint spanning trees in an edge-coloured K_n. By applying Theorem 2.1 to a large subgraph on vertices which have large remaining degree, and using a matching to extend the resulting tree to a spanning tree of K_n, we can find spanning rainbow trees with bounded maximum degree in which vertices with small degree in the remaining subgraph of K_n appear as leaves in each new rainbow spanning tree. This allows the iterative removal of spanning trees without decreasing the minimum degree too drastically. This iteration is carried out for the key lemma, Lemma 4.1.

The embedding differs depending on the colouring of K_n. To describe our cases, we use the following definitions.

Definition 2.2. Let $\alpha > 0$. We say a class of colours U is *α-rich* in an edge-coloured graph G if $\nu(G_U) \ge \alpha|G|$.

Note that when U consists of a single colour, being α-rich simply means that the maximal subgraph of G with edges in that colour contains a linearly-sized matching.

Definition 2.3. Let $\alpha > 0$. We say an edge-coloured graph G is *α-well-coloured* if its colours can be partitioned into $|G|-1$ α-rich classes.

Definition 2.4. Let $\alpha > 0$ and $t \in \mathbb{N}$. We say an edge-coloured graph G is *(α, t)-robustly-coloured* if, despite the removal of any t rainbow forests, G remains α-well-coloured.

If a graph does not have many rich colours, then it is robustly coloured, as follows.

Lemma 2.5. *Let $0 < \alpha \le 1/8$ and $\beta \le \alpha/4$. Then, for any properly edge-coloured K_n, either*
(i) *there are at least $n - 16\beta n$ colours that are α-rich, or*
(ii) *K_n is $(\beta, \beta n)$-robustly-coloured.*

Let K_n be properly edge-coloured. Letting $\alpha = 1/8$ and $\beta = \alpha/2400$, apply Lemma 2.5. If (ii) in Lemma 2.5 holds for K_n, then the following lemma shows that K_n contains at least $n/10^{12}$ edge-disjoint rainbow spanning trees.

Lemma 2.6. *Let $0 < \beta \le 1/2$. Then, every $(\beta, \beta n)$-robustly-coloured K_n contains at least $\beta^2 n/2500$ edge-disjoint rainbow spanning trees.*

If, however, (i) in Lemma 2.5 holds for K_n, then the following lemma shows that K_n contains at least $n/10^{12}$ edge-disjoint rainbow spanning trees.

Lemma 2.7. *Let $0 < \alpha \le 1/2$. Then, every properly edge-coloured K_n with at least $n - \alpha n/150$ α-rich colours contains at least $\alpha^2 n/10^6$ edge-disjoint rainbow spanning trees.*

Thus, in either case, K_n contains at least $n/10^{12}$ edge-disjoint rainbow spanning trees. Therefore, to prove Theorem 1.1 it is sufficient to prove Lemmas 2.5–2.7, which we do in the remaining three sections.