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Structural properties of resonance graphs of plane elementary bipartite graphs

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ABSTRACT

In this paper, we investigate some structural properties of resonance graphs of plane elementary bipartite graphs using Djoković – Winkler relation Θ and structural characterizations of a median graph. Let G be a plane elementary bipartite graph. It is known that its resonance graph $Z(G)$ is a median graph. We first provide properties for Θ-classes of the edge set of $Z(G)$. As a corollary, $Z(G)$ cannot be a nontrivial Cartesian product of median graphs, which is equivalent to a result given by Zhang et al. that the distributive lattice on the set of perfect matchings of G is irreducible. We then present a decomposition structure on $Z(G)$ with respect to a reducible face s of G. As an application, we give a necessary and sufficient condition on when $Z(G)$ can be obtained from $Z(H)$ by a peripheral convex expansion with respect to a reducible face s of G, where H is the subgraph of G obtained by removing all internal vertices (if exist) and edges on the common periphery of s and G. Furthermore, we show that $Z(G)$ can be obtained from $Z(H)$ by adding one pendent edge with the face-label s if and only if s is a forcing face of G such that both s and the infinite face of G are M-resonant for a degree-1 vertex M of $Z(G)$. Our results generalize the peripheral convex expansion structure on $Z(G)$ given by Klavžar et al. for the case when G is a catacondensed even ring system.

1. Introduction

The concept of resonance graphs was first introduced in chemistry [5,6], and was reintroduced in many other papers later, see [3,4,12,13]. A benzenoid graph (or, a hexagonal graph) is a 2-connected plane bipartite graph whose finite faces are regular hexagons of unit size. A benzenoid graph is called catacondensed if all vertices are located on the periphery of the graph. The concept of the resonance graph of a benzenoid graph was also given by Zhang et al. [18] under the name of Z-transformation graph, and was extended to that of a plane bipartite graph in [23]. Let G be a plane bipartite graph with a perfect matching. The Z-transformation graph (or, resonance graph) of G, denoted by $Z(G)$, is the graph whose vertices are the perfect matchings of G, and two vertices M_1 and M_2 of $Z(G)$ are adjacent if and only if their symmetric difference is the periphery of a finite face s of G, and we say that the edge M_1M_2 has the face-label s. It is well known [23] that if G is a plane elementary bipartite graph, then $Z(G)$ is a connected bipartite graph with at most two vertices of degree-1, and either is a path or a graph of girth 4 different from cycles. In [10], it was shown that if G is a plane weakly elementary bipartite graph, then the set $\mathcal{M}(G)$ of all perfect matchings of G is a finite distributive lattice and its Hasse diagram is isomorphic to the resonance digraph of G. By the lattice structure on $\mathcal{M}(G)$, Zhang et al. proved [20] that if G is a plane weakly elementary bipartite graph, then $Z(G)$ is a median graph. An important structure characterization of a median graph is the Mulder's convex expansion theorem: A graph is a median graph if and only if it can be obtained from the one vertex graph by a

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convex expansion procedure [7]. A peripheral expansion structure for the resonance graph of a catacondensed benzenoid graph was given in [8], and a peripheral convex expansion structure for the resonance graph of a catacondensed even ring system was given in [9]. The Djoković – Winkler relation Θ plays an important role on the structural characterization of median graphs. A characterization of the resonance graph of a catacondensed hexagonal graph was presented in [16] in terms of the induced graph on the Θ-classes of the edge set of the resonance graph. For more properties of resonance graphs, readers are recommended the survey paper on Z-transformation graphs of plane bipartite graphs by Zhang [19].

In Section 2, we introduce basic terminologies and known results that will be used in the paper. Let G be a plane elementary bipartite graph and $Z(G)$ be its resonance graph. In Section 3, we show that all edges of $Z(G)$ in a Θ-class have the same face-label. We then use it to prove that $Z(G)$ cannot be a nontrivial Cartesian product of median graphs, which is equivalent to a result given by Zhang et al. [22] that the distributive lattice $\mathcal{M}(G)$ on the set of perfect matchings of G is irreducible. A peripheral face s of G is called reducible if the subgraph H of G obtained by removing all internal vertices (if exist) and edges on the common periphery of s and G is a plane elementary bipartite graph. In Section 4, we provide a decomposition structure of $Z(G)$ with respect to a reducible face s of G. As an application, we give a necessary and sufficient condition on when $Z(G)$ can be obtained from $Z(H)$ by a peripheral convex expansion with respect to a reducible face s of G. This generalizes the results given in [8,9]. Furthermore, we show that $Z(G)$ can obtained from $Z(H)$ by adding one pendent edge with the face-label s if and only if s is a forcing face of G such that both s and the infinite face of G are M-resonant for a degree-1 vertex M of $Z(G)$.

2. Preliminaries

In this section, we introduce basic terminologies and known results that will be used in the paper. Let G be a graph. The vertex set of G is denoted by $V(G)$ and its edge set is denoted by $E(G)$. An induced subgraph of G generated by a subset $W \subseteq V(G)$, denoted by $\langle W \rangle$, is a graph with the vertex set W and two vertices are adjacent in $\langle W \rangle$ if and only if they are adjacent in G. We use uv to represent an edge of G with two end vertices u and v. The interval between two vertices u and v of G is the set of all vertices on all shortest paths between u and v in G, and denoted by $I_G(u, v)$. A median of three vertices u, v and w is a vertex that is contained in $I_G(u, v) \cap I_G(u, w) \cap I_G(v, w)$. A connected graph is called a median graph if every triple of its vertices has a unique median. An induced subgraph $\langle W \rangle$ of G is called a convex subgraph if the interval $I_G(u, v) \subseteq W$ for any $u, v \in W$. Let $d_G(u, v)$ denote the distance between two vertices u and v in G. If T is a subgraph of G such that $d_T(u, v) = d_G(u, v)$ for all $u, v \in V(T)$, then T is called an isometric subgraph of G.

A perfect matching (or, 1-factor) of a graph is a set of pairwise disjoint edges of the graph that cover all its vertices. A perfect matching of a benzenoid graph coincides with the Kekulé structure of the corresponding benzenoid hydrocarbon. Let M be a perfect matching of a graph G. An M-alternating cycle (resp., M-alternating path) of G is a cycle (resp., path) of G whose edges are alternately in and off M. We call that a path P of G is weakly M-augmenting if it satisfies one of the following conditions: (i) P has length 1 and the single edge of P is not contained in M, (ii) P is an M-alternating path such that its two end edges are not contained in M. Note that a weakly M-augmenting path is different from an M-augmenting path defined in [11], where the path has length > 1 and its two end vertices are not covered by the perfect matching M. Let M_1 and M_2 be two perfect matchings of G. Then a cycle of G is called (M_1, M_2)-alternating if its edges are in M_1 and M_2 alternately. The symmetric difference $A \oplus B$ of two sets A and B is the set of elements belonging to A or B but not both, that is, $A \oplus B = (A \cup B) \setminus (A \cap B)$. It is well known that the symmetric difference $M_1 \oplus M_2$ of two perfect matchings M_1 and M_2 of G is a union of disjoint (M_1, M_2)-alternating cycles of G.

A vertex of a plane graph is called an exterior vertex if it is located on the periphery (or, boundary) of the graph, and an interior vertex otherwise. Each interior region of a plane graph G is called a finite face of G, and the exterior region of G is called the infinite face (or, exterior face) of G. An even ring system is a 2-connected plane bipartite graph whose interior vertices are degree-3 vertices and exterior vertices are degree-2 or degree-3 vertices. A catacondensed even ring system is an even ring system such that all vertices are located on the periphery of the graph. A benzenoid graph (resp., a catacondensed benzenoid graph) is an even ring system (resp., a catacondensed even ring system) whose finite faces are regular hexagons of unit size. Two faces (one can be the infinite face) of a plane graph G are said to be adjacent if their peripheries have common edges. A finite face s of G is called a peripheral face if it is adjacent to the infinite face of G. If a plane graph G is 2-connected, then the periphery (or, boundary) of any face of G is a cycle. The periphery of the infinite face of G is denoted by ∂G, which is referred as the periphery of G. We use ∂s to represent the periphery of a finite face s of G. A face (including the infinite face) of G is called M-resonant if its periphery is an M-alternating cycle for a perfect matching M of G, and we say that a face is resonant briefly if there is no need to mention M. A perfect matching M of G is called a peripheral perfect matching (or, peripheral 1-factor) if the infinite face of G is M-resonant.

A bipartite graph G is called elementary if each edge is contained in some perfect matching of G. A plane bipartite graph G is elementary if and only if each face (including the infinite face) of G is resonant [23]. In particular, a benzenoid graph G is elementary if and only if the infinite face of G is resonant [17]. Elementary components of a bipartite graph G are the components obtained by removing all edges that are not contained in any perfect matchings of G. A plane bipartite graph G is called weakly elementary if every finite face of every elementary component of G is still a face of G. For example, benzenoid graphs are weakly elementary [23].

We assume that all vertices of a bipartite graph are colored white and black such that adjacent vertices cannot have the same color. A bipartite graph G is elementary if and only if it has a bipartite ear decomposition $G = e + P_1 + P_2 + \cdots + P_n$ [11]: