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Note

Sharp bounds for the Randić index of graphs with given minimum and maximum degree

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ABSTRACT

The Randić index of a graph G, written $R(G)$, is the sum of $\frac{1}{\sqrt{d(u)d(v)}}$ over all edges uv in $E(G)$. Let d and D be positive integers $d < D$. In this paper, we prove that if G is a graph with minimum degree d and maximum degree D, then $R(G) \geq \frac{\sqrt{dD}}{d+D}n$; equality holds only when G is an n-vertex (d, D)-biregular. Furthermore, we show that if G is an n-vertex connected graph with minimum degree d and maximum degree D, then $R(G) \leq \frac{n}{2} - \sum_{i=d}^{D-1} \frac{1}{2}\left(\frac{1}{\sqrt{i}} - \frac{1}{\sqrt{i+1}}\right)^2$; it is sharp for infinitely many n, and we characterize when equality holds in the bound.

1. Introduction

The *Randić index* of a graph G, written $R(G)$, is defined as follows:

$$R(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d(u)d(v)}},$$

where for a vertex $v \in V(G)$, $d(v)$ is the degree of v. The concept was introduced by Milan Randić under the name "*branching index*" or "*connectivity index*" in 1975 [18], which has a good correlation with several physicochemical properties of alkanes. In 1998 Bollobás and Erdös [5,6] generalized this index by replacing $-\frac{1}{2}$ with any real number α, which is called the general Randić index. There are also many other variants of Randić index [10,12,19]. For more results on Randić index, see the survey papers [14,17].

Many important mathematical properties of Randić index have been established. Especially, the relations between Randić index and other graph parameters have been widely studied, such as the minimum degree [5], the chromatic index [15], the diameter [10,20], the radius [8], the average distance [8], the eigenvalues [4,2], and the matching number [2].

In 1988, Shearer proved if G has no isolated vertices then $R(G) \geq \sqrt{|V(G)|}/2$ (see [11]). A few months later Alon improved this bound to $\sqrt{|V(G)|} - 8$ (see [11]). In 1998, Bollobás and Erdös [5] proved that the Randić index of an n-vertex graph G without isolated vertices is at least $\sqrt{n-1}$, with equality if and only if G is a star. In [11], Fajtlowicz mentioned that Bollobás and Erdös asked the minimum value for the Randić index in a graph with given minimum degree. Then the question was answered in various ways [1,9,13,16].

For a graph G, we denote its complement by $\overline{G}$, which is a graph with the same vertex set of G such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. We also denote by K_n the complete graph with n vertices

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and by $K_n - e$ the graph obtained from the complete graph K_n by deleting an edge. A graph is (a, b)-biregular if it is bipartite with the vertices of one part all having degree a and the others all having degree b.

Aouchiche et al. [3] studied the relations between Randić index and the minimum degree, the maximum degree, and the average degree, respectively. They proved that for any connected graph G on n vertices with minimum degree d and maximum degree D, then $R(G) \geq \frac{d}{d+D}n$.

In this paper, we prove that if G is an n-vertex graph with minimum degree d and maximum degree D, then $R(G) \geq \frac{\sqrt{dD}}{d+D}n$, which improves the result of Aouchiche et al. in [3]; equality holds only when G is an n-vertex (d, D)-biregular. Furthermore, we show that if G is an n-vertex connected graph with minimum degree d and maximum degree D, then $R(G) \leq \frac{n}{2} - \sum_{i=d}^{D-1} \frac{1}{2}\left(\frac{1}{\sqrt{i}} - \frac{1}{\sqrt{i+1}}\right)^2$; it is sharp for infinitely many n.

2. Main results

In this section, we first give a sharp lower bound for $R(G)$ in an n-vertex graph with given minimum and maximum degree, improving the one that Aouchiche et al. [3] proved.

Theorem 2.1. *If G is an n-vertex graph with minimum degree d and maximum degree D, then $R(G) \geq \frac{\sqrt{dD}}{d+D}n$. Equality holds only when G is an n-vertex (d, D)-biregular.*

Proof. For each $i \in \{d, \ldots, D\}$, let V_i be the set of vertices with degree i, and let $n_i = |V_i|$. Note that

$$\sum_{i=d}^{D} n_i = n. \tag{1}$$

Let $m_{ij} = |[V_i, V_j]|$ for all $i, j \in \{d, \ldots, D\}$, where $[A, B]$ is the set of edges with one end-vertex in A and the other in B. Since G has minimum degree d and maximum degree D, we have

$$R(G) = \sum_{d \leq i \leq j \leq D} \frac{m_{ij}}{\sqrt{ij}}. \tag{2}$$

For fixed i, the degree sum over all vertices in V_i can be computed by counting the edges between V_i and V_j over all $j \in \{d, \ldots, D\}$;

$$in_i = m_{ii} + \sum_{j=d}^{D} m_{ij}. \tag{3}$$

Note that m_{ii} must be counted twice.

By manipulating equation (3), we have the followings:

$$dn_d = (m_{dd} + \sum_{j=1}^{D} m_{dj}) \;\Rightarrow\; n_d - \frac{m_{dD}}{d} = \frac{1}{d}(m_{dd} + \sum_{j=d}^{D-1} m_{dj}) \tag{4}$$

$$Dn_D = (m_{DD} + \sum_{j=1}^{D} m_{Dj}) \;\Rightarrow\; n_D - \frac{m_{dD}}{D} = \frac{1}{D}(m_{DD} + \sum_{j=d+1}^{D} m_{jD}) \tag{5}$$

$$n_i = \frac{1}{i}(m_{ii} + \sum_{j=d}^{D} m_{ij}) \tag{6}$$

By Eqs. (1) and (6), we have

$$n_d + n_D = n - \sum_{i=d+1}^{D-1} n_i = n - \sum_{i=d+1}^{D-1} \frac{1}{i}(m_{ii} + \sum_{j=d}^{D} m_{ij}). \tag{7}$$

By combining Eqs. (4), (5), and (7), we have

$$n_d - \frac{m_{dD}}{d} + n_D - \frac{m_{dD}}{D} = n - \sum_{i=d+1}^{D-1} \frac{1}{i}(m_{ii} + \sum_{j=d}^{D} m_{ij}) - \left(\frac{d+D}{dD}\right) m_{dD}$$

$$= \frac{1}{d}(m_{dd} + \sum_{j=d}^{D-1} m_{dj}) + \frac{1}{D}(m_{DD} + \sum_{j=d+1}^{D} m_{jD}) \;\Rightarrow$$