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Note

Gender consistent resolving rules in marriage problems

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ABSTRACT

The selection of blocking pairs to be matched plays an important role in the study of mechanisms converting arbitrary matchings into stable ones. We assume that a resolving rule guides the selection and show that two axioms (independence and top optimality) transform such a rule into a gender consistent one. That is, the rule is forced by the axioms to follow a linear order over acceptable pairs which is consistent with the preferences of either all men or all women. As shown by Abeledo and Rothblum (1995), stable matchings can be reached when starting from an arbitrary individually rational matching and iteratively satisfying the pair selected by a gender consistent resolving rule.

1. Introduction

The selection of blocking pairs to be matched plays an important role in the study of mechanisms converting arbitrary matchings into stable ones as shown in the seminal works of Knuth [4] and Roth and Vande Vate [9]. The latter work considers sequences of matchings, where each matching is obtained from the previous one by satisfying a single blocking pair, and introduces an algorithm which selects in a specific way the blocking pair to be satisfied at each iteration until a stable matching is reached.[1] The focus on matching a single blocking pair at each iteration of an algorithm turned out to be fruitful also with respect to the original Gale–Shapley procedure (see, e.g., [2]) for showing the existence of a stable matching. For instance, McVitie and Wilson [7] have modified Gale and Shapley's algorithm by letting at each iteration only one man propose to the woman he prefers most among those who have not yet rejected him.

The single-proposal variant of the Gale–Shapley algorithm and the Roth–Vande Vate algorithm were shown in [1] to share a common feature. More precisely, the common feature is shared by the corresponding *resolving rules*, that is, by the rules applied in both algorithms for the selection of a blocking pair to be matched from the set of all pairs blocking a given matching. These resolving rules turn out to follow a pre-specified *linear order* over the set of acceptable pairs which is *gender consistent*, that is, the linear order is consistent with the preferences of either all men or all women (see Sections 2 and 3 for the precise definitions).

In the current paper we provide an axiomatic foundation of gender consistent resolving rules. In doing so, we focus on male consistency, that is, the corresponding rule shall respect the preferences of all men. When defining the domain on which a resolving rule is supposed to work, we consider preference profiles differing only with respect to women's preferences and matchings that are individually rational (but not stable) at such profiles. Given such a domain, a resolving rule then assigns, to each preference profile and matching, a pair that blocks the matching at that preference profile. It is worth mentioning that the above domain restrictions are in line with Abeledo and Rothblum [1], where male consistency is defined with respect to a fixed profile of men's preferences and matchings are always supposed to be individually rational.

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[1] As shown by Ma [6], the mechanism suggested by Roth and Vande Vate [9] does not always reach all stable matchings.

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The algorithms based on gender consistent resolving rules can generally be interpreted as describing different ways for assigning a stable matching to any (individually rational) initial matching. In that sense, our work can be seen in the same vain as the axiomatically driven papers of Kojima and Manea [5] and Morrill [8]. The rules characterized in these papers are based on the deferred acceptance algorithm applied in the context of object allocation problems. In contrast, and to the best of our knowledge, the present paper is the first one to offer an axiomatic characterization of a whole class of rules in the context of two-sided matching.

We formulate two requirements – an independence axiom and a top man optimality condition – and impose them on a resolving rule. The independence axiom works on different pairs consisting of a preference profile and a matching, and forces a resolving rule to select the same outcome, provided that the corresponding sets of blocking pairs do coincide. As for our second requirement, we slightly modify the notion of domination among blocking pairs introduced in Klijn and Massó [3], and require in the top man optimality condition any resolving rule outcome to be undominated.

The rest of the paper is organized as follows. Section 2 contains basic notation and definitions, as well as the precise formulations of the proposed axioms. We then show in Section 3 that a resolving rule satisfying these axioms must be male consistent. Finally, in Section 4, we embed two stable matching problems in our framework and discuss matching sequences that cycle when the resolving rule applied satisfies one of the axioms but not the other.

2. Setup and axioms

We consider the standard two-sided one-to-one matching model introduced in Gale and Shapley [2], in which there are two finite sets M and W of agents, called "men" and "women", respectively. These two sets will be kept fixed throughout the paper with each agent being endowed with a complete, transitive, and antisymmetric binary relation over the agents from the opposite sex and the possibility of remaining single. For instance, $w \succ_m w'$ expresses the fact that man $m \in M$ prefers woman $w \in W$ to woman $w' \in W$; we write $w \succeq_m w'$ whenever either $w \succ_m w'$ or $w = w'$ holds. A preference profile is denoted by $\succeq = (\succeq_i)_{i \in M \cup W}$, and we say that a pair $(m, w) \in M \times W$ is *acceptable* at $\succeq$ if $w \succ_m m$ and $m \succ_w w$ hold. The set of all acceptable pairs at $\succeq$ is denoted by $A^{\succeq}$.

A *matching* μ is a function $\mu : M \cup W \to M \cup W$ such that $\mu(m) \in W \cup \{m\}$, $\mu(w) \in M \cup \{w\}$, and $\mu^2(i) = i$ hold for $m \in M$, $w \in W$, and $i \in M \cup W$. The interpretation of $\mu(i) = i$ for some $i \in M \cup W$ is that the corresponding agent is single under μ. If $\mu(i) = i$ holds for each $i \in M \cup W$, we say that the corresponding matching is *empty* and denote it throughout the paper by μ_0. A matching μ is called *stable* at $\succeq$ if (1) it consists of either singletons or acceptable pairs at $\succeq$ (*individual rationality*), and (2) there are no blocking pairs at $\succeq$ for it, i.e., there is no pair (m, w) such that $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$. The set of blocking pairs for a matching μ at a profile $\succeq$ is denoted by $B^{\succeq}(\mu)$.

Let $\mathscr{M}$ stand for the set of all matchings, $(\succeq_m)_{m \in M}$ be a fixed profile of men's preferences, and $\mathscr{R}$ be the largest collection of preference profiles for which $\succeq' \in \mathscr{R}$ implies $\succeq'_m = \succeq_m$ for each $m \in M$. We collect in the set $\mathscr{D}$ all $(\succeq, \mu) \in \mathscr{R} \times \mathscr{M}$ such that μ is individually rational but not stable at $\succeq$. A *resolving rule* g is defined on $\mathscr{D}$ and assigns a pair $(m, w) \in B^{\succeq}(\mu)$ to each $(\succeq, \mu) \in \mathscr{D}$. Notice then that, for each $(\succeq, \mu) \in \mathscr{D}$, we have $B^{\succeq}(\mu) \subseteq A^{\succeq}$.

The first axiom we impose on a resolving rule is a standard independence requirement. In our context it captures the idea that, when comparing two different pairs of preference profiles and matchings, the outcome of a resolving rule remains the same if the corresponding sets of blocking pairs are related by set inclusion as specified below. As a consequence, the axiom allows us to conclude that it is the *set* of blocking pairs that matters when a rule satisfying the axiom selects a particular pair. Notice additionally that the corresponding set of blocking pairs is well defined and non-empty for any matching and preference profile in the domain we consider. Thus, a resolving rule can in fact be seen as selecting a single blocking pair from a set of blocking pairs; our independence requirement correctly supports this particular interpretation.

Independence of Irrelevant Alternatives (IIA): If $B^{\succeq}(\mu) \subseteq B^{\succeq'}(\mu')$ for some $(\succeq, \mu), (\succeq', \mu') \in \mathscr{D}$, then $g(\succeq', \mu') \in B^{\succeq}(\mu)$ implies $g(\succeq, \mu) = g(\succeq', \mu')$.

In order to introduce our second axiom, let us consider $(\succeq, \mu) \in \mathscr{D}$ with $(m, w), (m, w') \in B^{\succeq}(\mu)$. Moreover, let man m be the top man for both w and w' at $\succeq$; that is, $m \succeq_{w''} m''$ holds for $w'' \in \{w, w'\}$ and all $m'' \in M$. We deem then g *top man dominated* at $(\succeq, \mu)$ if $g(\succeq, \mu) = (m, w)$ and $w' \succ_m w$. A resolving rule g is *top man optimal* if it is not top man dominated at any $(\succeq, \mu) \in \mathscr{D}$.

Top Man Optimality (TMO): g is top man optimal.

As to understand the power of TMO, suppose that $g(\succeq, \mu) = (m, w)$ holds for some $(\succeq, \mu) \in \mathscr{D}$. Notice then that g being top man optimal does not necessarily imply that m is a top man at $\succeq$ for w. It rather says that, *provided* m *is* the top man for w at $\succeq$, there is no other woman w' *with the same top man* at $\succeq$ such that $(m, w') \in B^{\succeq}(\mu)$ and $w' \succ_m w$.[2]

[2] Klijn and Massó [3] introduce weak stability for the marriage model that allows for the existence of weak blocking pairs, and show that Zhou's bargaining set [10] for their context does coincide with the set of weakly stable and weakly efficient matchings. More precisely, a pair $(m, w) \in B^{\succeq}(\mu)$ is called *weak* if there exists either a woman w' with $(m, w') \in B^{\succeq}(\mu)$ and $w' \succ_m w$, or a man m' with $(m', w) \in B^{\succeq}(\mu)$ and $m' \succ_w m$. Notice that our dominance notion is stronger and thus, the corresponding TMO condition allows for the selection of weak blocking pairs.

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