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Conditional edge-fault-tolerant Hamiltonicity of the data center network

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ABSTRACT

The k-dimensional data center network with n port switches, denoted by $D_{k,n}$, has been proposed for data centers as a server centric network structure. Wang et al. (2015) had shown that $D_{k,n}$ is $(n+k-3)$-fault-tolerant Hamiltonian. In this paper, we consider more faulty edges and prove that $D_{k,n}$ is conditional $(2n+2k-9)$-edge-fault-tolerant Hamiltonian for any $k \geq 0$ and $n \geq 2$ except $k=1$ and $n \geq 6$. Moreover, the upper bound $2n+2k-9$ of $|F|$ is optimal.

1. Introduction

It is well known that an interconnection network can be modeled by a loopless undirected graph $G=(V,E)$, where V is the set of processors and E is the set of communication links. In this paper, we use graphs and networks interchangeably. Since failures may occur when a network is put into use, it is practically meaningful and important to consider networks with faulty edges or faulty nodes. There are major two faulty assumptions, one is the random faulty assumption, which posits that there is no restriction on the distribution of faulty edges; another is the conditional faulty assumption, which posits that each node is incident with at least two fault-free edges. Many researchers consider the random faulty model for various networks, for example [2,4–6,11,14,23,24].

There are many applications of Hamiltonian cycles and paths to practical problems including on-line optimization of a complex flexible manufacturing system in [1] and wormhole routing in [17,22]. With the increase of the processors and the complexity of large-scale processing systems, it is almost impossible for a processor being isolated (all links incident to it are faulty) or pendent (only one link incident to it is fault-free and others are all faulty) in reality. It motivates the researches on the conditional edge-fault-tolerant Hamiltonicity under the assumption each vertex is incident with at least two edges which are not faulty. There are rich results about conditional edge-fault-tolerant Hamiltonicity of networks. For examples, Hsieh et al. [12,13] proved the existence of a fault-free Hamiltonian cycle in star graph S_n and locally twisted cubes with $3n-10$ and $2n-5$ conditional faulty edges, respectively. Fu [8] and Chen et al. [3] studied the edge-fault-tolerant Hamiltonicity of the twisted cube and the dual-cube DC_n with $2n-5$ and $2n-3$ conditional faulty edges, respectively.

Concurrently, a server-centric data center network called *DCell* is proposed by Guo et al. in [10] which addresses the needs of mega data centers and it has many desired properties including doubly exponential, high network capacity, small diameter and high fault tolerance. In [15], Kliegl et al. presented generalized DCell graphs including DCell satisfying all of the good properties above. The generalized DCell graphs provide much better loading-balancing property than DCell. Gu et al. [9] obtained the pessimistic diagnosability of DCell. Since finding a Hamiltonian cycle or path is NP-complete, some

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researches on Hamiltonicity focus on the data center networks. Wang et al. solved the problem of one-to-one disjoint path covers of DCell in [19], constructed $n+k-1$ vertex–disjoint paths between every two distinct vertices of DCell in [20] and gave the restricted h-connectivity of DCell in [21]. Tian et al. studied the vertex-pancyclicity of DCell in [16]. In [18], Wang et al. showed that (1) a k-dimensional DCell with n port switches is Hamiltonian-connected for $k \geq 0$ and $n \geq 2$, except for $(k, n) = (1, 2)$; (2) a k-dimensional generalized DCell with n port switches is Hamiltonian-connected for $k \geq 0$ and $n \geq 3$; (3) a k-dimensional DCell is $(n+k-4)$-fault-tolerant Hamiltonian-connected and is $(n+k-3)$-fault-tolerant Hamiltonian. All of these results demonstrate that $D_{k,n}$ is excellent in communication.

In this paper, we consider the edge-fault-tolerant Hamiltonicity of DCell with more faulty edges. Let F be any conditional edge-fault-set, we prove that $D_{k,n}$ is conditional $(2n+2k-9)$-edge-fault-tolerant Hamiltonian for any $k \geq 0$ and $n \geq 2$ except $k = 1$ and $n \geq 6$. Let $[n] = \{1, 2, \ldots, n\}$ and $\langle n \rangle = \{0, 1, 2, \ldots, n\}$.

The remainder of this paper is organized as follows. Section 2 introduces necessary definitions and notations. Our main result, the conditional edge-fault-tolerant Hamiltonicity of the data center network, is shown in Section 3.

2. Definitions and preliminaries

Let $G = (V, E)$ be a simple undirected graph. Two vertices v_1, v_2 in V are said to be *adjacent* if and only if $(v_1, v_2) \in E$ and v_1, v_2 are said to be *incident* with the edge (v_1, v_2), and vise versa. Two edges which are incident with a common vertex are also said to be *adjacent*. The *neighbor* of a vertex u in G is a vertex adjacent to u in V and the *neighborhood* of a vertex u in G is a set of all neighbors of u, denoted by $N_G(u)$. The cardinality $|N_G(u)|$ represents the *degree* of u in G, denoted by $d_G(u)$ (or simply $d(u)$), $\delta(G)$ is the *minimum degree* of G. Suppose that V' is a nonempty subset of V and E' is a nonempty subset of E, then $G - V'$ is the subgraph obtained from G by deleting the vertices in V' together with their incident edges, if $V' = \{v\}$, we write $G - v$ for $G - \{v\}$; and $G - E'$ is the subgraph obtained from G by deleting the edges in E'. Similarly, the graph obtained from G by adding a set of edges in E' is denoted by $G + E'$, if $E' = \{e\}$, we write $G - e$ and $G + e$ instead of $G - \{e\}$ and $G + \{e\}$. A *path* $P = (v_1, v_2, \ldots, v_k)$ for $k \geq 2$ in G is a sequence of distinct vertices such that any two consecutive vertices are adjacent. The two vertices v_1 and v_k are the *end-vertices* (or simply *ends*) of the path P, we can also call P is a (v_1, v_k)-*path*. A path $P = (v_1, v_2, \ldots, v_k)$ forms a *cycle* C if $v_1 = v_k$ and $k \geq 3$. The number of edges of a path P (a cycle C) is its *length*, denoted by $|P|(|C|)$. The *connectivity* $\kappa(G)$ of a connected graph G is the minimum number of vertices removed to get the graph either disconnected or trivial.

For a graph $G = (V, E)$, it is *Hamiltonian* if there exists a *Hamiltonian cycle* which is a cycle containing all the vertices of the graph G and it is *Hamiltonian-connected* if there exists a *Hamiltonian path* which is a path containing all the vertices of a graph G connecting any pair of vertices. Moreover, it is *k-fault-tolerant Hamiltonian* (respectively, *k-fault-tolerant Hamiltonian-connected*) if there exists a Hamiltonian cycle (respectively, if each pair of vertices can be joined by a Hamiltonian path) in $G \setminus F$ for any set F of elements (vertices and/or edges) with $|F| \leq k$. We called the set F of elements (vertices and/or edges) which are deleted in G *fault-set*. If F is the set of edges, then F is an *edge-fault-set*. Let F be an edge-fault-set of G, if $\delta(G \setminus F) \geq 2$, then F is said to be a *conditional edge-fault-set*. A graph G is *conditional k-edge-fault-tolerant Hamiltonian* if the resulting graph after removing any conditional edge-fault-set with at most k faulty edges contains a Hamiltonian cycle. A *complete graph* of n vertices, denoted by K_n, is a simple graph whose vertices are pairwise adjacent.

Definition 1 (*[10]*)**.** Let $D_{k,n}$ denote a k-dimensional data center network with n port switches for each $k \geq 0$ and $n \geq 2$. (The data center network is simple say DCell.) And $D_{k,n}$ can be defined recursively as following. Let $D_{0,n}$ be a complete graph K_n and $t_{k,n}$ be the number of the vertices in $D_{k,n}$ (so, $t_{0,n} = n$).

For $k > 0$, $D_{k,n}$ is built from $t_{k-1,n} + 1$ disjoint copies of $D_{k-1,n}$, where $D^i_{k-1,n}$ denotes the ith copy. Each pair of $D_{k-1,n}$, say $D^a_{k-1,n}$ and $D^b_{k-1,n}$, is connected by a unique *k-dimensional edge*, say (x, y), according to the following **Connection rule**. If (x, y) is a k-dimensional edge, then y is called the (unique) *k-dimensional neighbor* of x, usually say $y = x^k$ (or $x = y^k$). The only edge (x, y) between $D^a_{k-1,n}$ and $D^b_{k-1,n}$ is also denoted by $(D^a_{k-1,n}, D^b_{k-1,n})$.

A vertex v in $D^a_{k-1,n}$ is labeled by a $(k+1)$-tuple $(v_k, v_{k-1}, \ldots, v_0)$, where $v_i < t_{i-1,n} + 1$ $(0 < i \leq k)$, $v_k = a$ and $v_0 < n$. The *suffix* $(v_j, v_{j-1}, \ldots, v_0)$ has the unique $uid_j(v)$, given by $uid_j(v) = v_0 + \sum_{l=1}^{j}(v_l \times t_{l-1,n})$. In $D_{k,n}$, each vertex is uniquely identified by its $(k+1)$-tuple and the $(k+1)$-tuple can also be derived from its unique uid_k.

Connection rule: For each pair of $D_{k-1,n}$, say $D^a_{k-1,n}$ and $D^b_{k-1,n}$ $(a < b)$, the vertex $u = (u_k, u_{k-1}, \ldots, u_0)$ in $D^a_{k-1,n}$ is incident with the vertex $v = (v_k, v_{k-1}, \ldots, v_0)$ in $D^b_{k-1,n}$ if and only if $u_k = a = uid_{k-1}(v)$ and $v_k = b = uid_{k-1}(u) + 1$.

Fig. 1 depicts some $D_{k,n}$ with small parameters n and k. For convenience, in $D_{k,n}$ $(k \geq 1)$, we put D_i shorthand for $D^i_{k-1,n}$ and use $\overline{D_{k,n}}$ to denote the graph obtained by contracting each D_i to a vertex i and the edges of $\overline{D_{k,n}}$ are corresponding to all k-dimension edges in $D_{k,n}$. By Definition 1, $\overline{D_{k,n}} \cong K_{t_{k-1,n}+1}$ $(k \geq 1)$.

Definition 2. Let F be an edge-fault-set in $D_{k,n}$ $(k \geq 1)$, a path $\overline{P} = (x_1, x_2, \ldots, x_k)$ (a cycle $\overline{C} = (x_1, x_2, \ldots, x_k)$ where $x_1 = x_k$) in $\overline{D_{k,n}}$ is fault-free if any edge in $D_{k,n}$ corresponding to the edge $(x_i, x_{i+1}) \in E(\overline{P})$ ($(x_i, x_{i+1}) \in E(\overline{C})$) for $1 \leq i \leq k-1$ is fault-free.

Lemma 2.1 ([7])**.** *Suppose F is any conditional edge-fault-set in a complete graph K_n where $n \geq 4$. For $n \notin \{7, 9\}$ (respectively, $n \in \{7, 9\}$), if $|F| \leq 2n-8$ (respectively, $|F| \leq 2n-9$), then $K_n - F$ is Hamiltonian.*