ARTICLE IN PRESS
Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Extending single tolerances to set tolerances

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ARTICLE INFO

Article history:
Received 27 March 2017
Received in revised form 19 March 2018
Accepted 21 March 2018
Available online xxxx

Keywords:
Sensitivity analysis
Combinatorial optimization
Single tolerance
Set tolerance

ABSTRACT

The theory of single upper and lower tolerances for combinatorial minimization problems was formalized in 2005 for the three types of cost functions sum, product, and maximum, and since then it has shown to be rather useful in creating heuristics and exact algorithms. However, such single tolerances are often used because the assessment of multiple cost changes is considered too complicated. This paper addresses that issue. In this paper we extend this theory from single to set tolerances for these three types of cost functions. In particular, we characterize specific values of set upper and lower tolerances as positive and infinite, and we show a criterion for the uniqueness of an optimal solution to a combinatorial minimization problem. Furthermore, we present one exact formula and several bounds for computing set upper and lower tolerances using the relation to their corresponding single tolerance counterparts.

1. Introduction

The notion of tolerances originates from *sensitivity analysis* of combinatorial minimization problems [7,8,15,31], which is a well-established topic in linear programming [8] and mixed integer programming [15]. The notion of *single tolerance* corresponds to the most elementary topic of sensitivity analysis, namely the special case when the value of a single element in a feasible solution is subject to an additive change. More precisely, for an element *in* a given optimal solution, its *single upper tolerance* determines the maximum additive increase of the individual cost of this given element preserving the optimality of this solution, while keeping the costs of other elements unchanged. Analogously, for an element *not in* a given optimal solution, its *single lower tolerance* determines the maximum additive decrease of the individual cost of this given element preserving the optimality of this solution, while keeping the costs of other elements unchanged. So the tolerance is a measure of stability of optimal solutions. Efficient methods for computing (single) tolerances have been presented for the following combinatorial minimization problems: for the Minimum Spanning Tree Problem (MSTP) [5,19,32], the TSP [23], the Linear Assignment Problem (LAP) [36], network flow problems [16,29], shortest path problems [26,29], scheduling problems [18], the Maximum Capacity Problem [26], and linear forms [30]. The first successful implicit application of (upper) tolerances in algorithm design has appeared in the so-called Vogel's Approximation Method for the Transportation Simplex Problem [27]. Furthermore, tolerances have been used for a straightforward enumeration of the k-best solutions for some natural k for the Linear Assignment Problem [25] and the TSP [34] as well as a base of the Max-Regret heuristic for solving the Three-Index Assignment Problem [1].

The theory of single tolerances has been formalized by Goldengorin, Jäger, Molitor [11,13] for three different types of cost functions, namely of types sum, product and maximum. Based on this theory, effective heuristics and exact algorithms

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https://doi.org/10.1016/j.dam.2018.03.053

have been created and implemented [6,9,10,12,14,20,28,33] for the TSP as well as for related problems [2,17,22], proving the usefulness of the concept of tolerances. In 2010, Libura introduced and investigated the so-called *robustness tolerance* [24] where one not only determines for which changes a solution remains optimal, but also where it remains robust, i.e., gives the lowest regret. However, this is not a topic of this paper.

One disadvantage of using single tolerance values is that they only apply to changes in single parameter values. We cannot consider the effect of multiple parameter changes. Such multiple parameter changes are relevant in sensitivity analysis, as we illustrate in Section 3, but also in the above mentioned algorithms. Currently, these algorithms consider the deletion or inclusion of one element at a time. However, if we know that multiple elements are to be removed (included), a tool that computes the joint effect of the removal (inclusion) of multiple elements would be very useful indeed.

The purpose of this work is to extend the theory of single tolerances to so-called *set tolerances* where the set upper tolerances are defined for a set of elements *in* a given optimal solution, and the set lower tolerances are defined for a set of elements *not in* a given optimal solution. The *set tolerance* is defined as the maximum sum of values that can be added to the elements of the set so that the given optimal solution stays optimal.

The main question in this paper is whether and how set tolerances can be computed using single tolerances and the values they can attain.

We reach the following results where some of them are only valid for some of the three types of cost functions:

- The set upper and lower tolerances are well defined, i.e., do not depend on the corresponding optimal solution.
- The sets overlapping with each feasible solution are exactly the sets with infinite set upper tolerance.
- The sets not contained in the union of all feasible solutions are exactly the sets with infinite set lower tolerance.
- The uniqueness of an optimal solution can be described by the set upper and lower tolerances.
- The set upper and lower tolerances can be bounded by their corresponding single tolerance counterparts (i.e., the single tolerances of all elements in the set). The relations are completely different for the three types of cost function.

Some of the theorems in this paper were presented in a previous work [21], but proofs and interpretations of the results were not presented.

This paper is organized as follows. In Section 2, we give the notions of combinatorial minimization problem and of the single upper and lower tolerances. In Section 3, we highlight the shortcomings of single tolerances and motivate the usage of set tolerances. In Section 4, we present the theory of set upper tolerances and in Section 5, we proceed with the theory of set lower tolerances. In Section 6, we provide several basic computational examples and in Section 7, we give two applications of this theory, namely the Linear Assignment Problem and the Asymmetric Bottleneck Traveling Salesman Problem. Finally, Section 8 provides the conclusions and some suggestions for future research.

2. Notations and definitions

In this section, we formally present the notation and key results on combinatorial minimization problems and single upper and lower tolerances (see [11,13]). Note that we add the adjective 'single' to distinguish these from set tolerances.

2.1. Combinatorial minimization problems

A *combinatorial minimization problem* $\mathcal{P}$ is given by a tuple $(\mathcal{E}, D, c, f_c)$ where $\mathcal{E}$ is a finite *ground set of elements*, $D \subseteq 2^{\mathcal{E}} \setminus \{\emptyset\}$ is the set of *feasible solutions*, $c : \mathcal{E} \to \mathbb{R}$ is the *cost function*, which assigns costs to each single element of $\mathcal{E}$, $f_c : D \to \mathbb{R}$ is the objective (cost) function, which depends on the function c and assigns costs to each feasible solution D.

Then the problem is to find a feasible solution with a cost as small as possible. Of course, analogous considerations can be made if the costs have to be maximized, i.e., for combinatorial maximization problems.

$S^\star \subseteq \mathcal{E}$ is called an *optimal solution* of $\mathcal{P}$ if $S^\star$ is a feasible solution and the cost $f_c(S^\star)$ of $S^\star$ is minimum, i.e., $S^\star \in D$ and $f_c(S^\star) = \min \{f_c(S) \mid S \in D\}$. We denote the cost of an optimal solution $S^\star$ of $\mathcal{P}$ by $f_c(\mathcal{P})$ and the set of optimal solutions by $D^\star$. There are some particular cost functions which often occur in practice, namely (cf. Examples 1 and 2):

- The cost function $f_c : D \to \mathbb{R}$ is of **type** $\sum$ if for all $S \in D : f_c(S) = \sum_{e \in S} c(e)$ holds.
- The cost function $f_c : D \to \mathbb{R}$ is of **type** $\prod$ if for all $S \in D : f_c(S) = \prod_{e \in S} c(e)$ holds and for all $e \in \mathcal{E} : c(e) > 0$ holds.
- The cost function $f_c : D \to \mathbb{R}$ is of **type** MAX if for all $S \in D : f_c(S) = \max \{c(e) \mid e \in S\}$ holds. Such a cost function is also called *bottleneck function*.

Cost functions of type $\sum$, $\prod$, MAX are *monotonically increasing* in a single element $e \in \mathcal{E}$, i.e., the cost of a subset of $\mathcal{E}$ does not become cheaper, if the cost of e increases.

Furthermore, cost functions of type $\sum$, $\prod$, MAX are *continuous* when changing cost values. As in [11,13], we only consider combinatorial minimization problems $\mathcal{P} = (\mathcal{E}, D, c, f_c)$ that fulfill the following three conditions:

Condition 1. *The set D of feasible solutions of $\mathcal{P}$ is independent of the cost function c.*

Condition 2. *The cost function $f_c : D \to \mathbb{R}$ is of type $\sum$, $\prod$, or MAX.*

Condition 3. *There is at least one optimal solution of $\mathcal{P}$, i.e., $D^\star \neq \emptyset$.*

Please cite this article in press as: G. Jäger, M. Turkensteen, Extending single tolerances to set tolerances, Discrete Applied Mathematics (2018), https://doi.org/10.1016/j.dam.2018.03.053.