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On optimal piercing of a square

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ABSTRACT

We treat the following problem: given an $n \times n$ square $ABCD$, determine the minimum number of points that need to be chosen inside the square $ABCD$ such that there does not exist a unit square inside the square $ABCD$ containing none of the chosen points in its interior. In other words, we are interested to know how to most efficiently "destroy" a square-shaped object of side length n, where "destroying" is achieved by piercing as few as possible small holes, and the square is considered "destroyed" if no unpierced square piece of unit side length can be salvaged. This problem actually belongs to the family of problems centered about the so-called *piercing number*: indeed, if $\mathscr{U}_n$ denotes the collection of all open unit squares that can be fitted inside a given $n \times n$ square, the value that we are looking for is the piercing number of the collection $\mathscr{U}_n$, denoted by $\pi(\mathscr{U}_n)$. We show that $\pi(\mathscr{U}_n) = n^2$ when $n \leqslant 7$, and give an upper bound for $\pi(\mathscr{U}_n)$ that is asymptotically equal to $\frac{2}{\sqrt{3}}n^2$, which we believe is asymptotically tight. We then generalize our reasoning in order to obtain a similar upper bound when $ABCD$ is a rectangle, as well as an upper bound for $\pi(\mathscr{U}_x)$ when x is not necessarily an integer. Finally, we show that our results have an application to the problem of packing a given number of unit squares in the smallest possible square; it turns out that our results present a general "framework" based on which we are able to reprove many results on the mentioned problem (originally obtained independently of each other) and also obtain a new result on packing 61 unit squares.

1. Introduction

There are several lines of research concerning arrangements of unit squares with respect to a larger square, such as packing n unit squares in the smallest possible square [11], or covering the largest possible square with n unit squares [12]. There are also several lines of research concerning arrangements of points inside a given square, such as the problem initiated by Moser [18] to find how large the minimum distance determined by n points in a unit square can be (which is today often researched in its equivalent form of packing circles in a square [3, Section D1] [23]) or the problem of determining the area of the largest convex region not containing in its interior any of n points chosen in a unit square [19,21].

We hereby study a problem that presents a kind of interplay between these two classes of problems. In fact, it belongs to a (quite general) family of problems centered about the so-called *piercing number*. Namely, given a collection of figures $\mathscr{F}$ in the Euclidean plane (or, more generally, space), the *piercing number* of $\mathscr{F}$, denoted by $\pi(\mathscr{F})$, is defined as the minimum number of points that need to be chosen in such a way that each figure from $\mathscr{F}$ contains at least one of the chosen points (in other words, how many "needles" are required to pierce all members of $\mathscr{F}$). One of the first questions of this kind was asked by Gallai [10, Section III.13]: determine the smallest integer k such that, given any family of circular disks in the plane where every two of them have a common point, there exists a set of k points such that each disk contains at least one of those

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points; in other words, the value that is asked for equals $\sup_{\mathscr{F}} \pi(\mathscr{F})$, where $\mathscr{F}$ ranges over all the described families of disks. (It is now known that the answer is $k = 4$, where the lower bound is due to Grünbaum [14], while a proof of the upper bound had been announced by Danzer in 1954, though the first published proof is due to Stachó [22]; Danzer himself published [5] a proof in 1986, though this is not his original proof.) Various other problems of this kind have been investigated: when the space is d-dimensional, when all the disks are congruent, when the family consists of translates/homothetic images of a given (usually convex) figure etc. (We mention, for example, the result of Karasev [16], who proved that 3 points are always sufficient, and sometimes necessary, to pierce any family of translates of a compact convex set in the plane, any two of which have nonempty intersection.) These problems are usually called Gallai-type problems. A further family of problems that has attracted quite a lot of attention is the family of the so-called (p, q)-problems. They ask for piercing numbers of finite families of sets in the d-dimensional space, such that among every p members of the family there exist q of them with a nonempty intersection. One of the most important results for this class of problems is proved by Alon and Kleitman [1], who showed that, whenever p, q and d are fixed and $p \geqslant q \geqslant d + 1$, then $\pi(\mathscr{F})$ has an upper bound depending only on (p, q, d). However, exact values of $\sup_{\mathscr{F}} \pi(\mathscr{F})$ (for fixed p, q, d) are known only in some very special cases. Apart from $d = 1$, when it is known that $p + q - 1$ are always sufficient and sometimes necessary to pierce $\mathscr{F}$ (proved by Hadwiger and Debrunner [15], in the paper where this family of problems has actually been introduced), even for $(p, q, d) = (4, 3, 2)$ it is only known that the supremum is bounded below by 3 (see [4]) and above by 13 (see [17]). For more information about problems related to the piercing number, see the surveys [6–8].

We treat the following problem: given an $n \times n$ square $ABCD$, determine the minimum number of points that need to be chosen inside the square $ABCD$ such that there does not exist a unit square inside the square $ABCD$ containing none of the chosen points in its interior. In other words, if $\mathscr{U}_n$ denotes the collection of all open unit squares that can be fitted inside a given $n \times n$ square, we are looking for the value $\pi(\mathscr{U}_n)$. The problem can also be presented in the following way: we are interested to know how to most efficiently "destroy" a square-shaped object of side length n, where "destroying" is achieved by piercing as few as possible small holes, and the square is considered "destroyed" if no unpierced square piece of unit side length can be salvaged. Stated like this, it seems that this problem is quite applicable in real life. Furthermore, as it will turn out, it also has a direct application to the already mentioned research problem of packing n unit squares in the smallest possible square.

The work is divided into sections as follows. In Section 2 we show that for $n \in \mathbb{N}, n \leqslant 4$, we have $\pi(\mathscr{U}_n) = n^2$ (note that n^2 is a trivial lower bound for $\pi(\mathscr{U}_n)$, and thus we only need to prove that $\pi(\mathscr{U}_n) \leqslant n^2$). In Section 3 we prove an upper bound for $\pi(\mathscr{U}_n)$ asymptotically equal to $\frac{2}{\sqrt{3}}n^2$. Our upper bound actually matches the lower bound n^2 for $n \leqslant 7$, and thus we get a corollary that for $n \leqslant 7$ we have $\pi(\mathscr{U}_n) = n^2$. (This in fact includes the results from Section 2 as a special case. However, in Section 3 we actually use some parts of the proof from Section 2, while the construction given in Section 2 is much more natural and thus we believe that the underlying idea is simpler to understand if seen on that construction first.) In Section 4 we show how the upper bound from Section 3 can be easily generalized to the case when $ABCD$ is a rectangle; we then modify the upper bound from Section 3 in order to obtain an upper bound for $\pi(\mathscr{U}_x)$ when x is not necessarily an integer. In Section 5 we show that our results enable us to reproduce, as a direct consequence, some known results on the square packing problem (among which is a result that the smallest square in which 46 unit squares can be packed is the square of side length 7, which has been proved only recently [2]), and further obtain a new result on packing 61 unit squares. Finally, in Section 6 we state a conjecture about asymptotical tightness of our upper bound for $\pi(\mathscr{U}_n)$.

Our techniques remind of some ideas often used in the context of "unavoidable points", a notion developed by Friedman [11] in relation to the square packing problem; in fact, some of our proofs can be a little bit shortened by appealing to some lemmas from there. We instead choose to write the paper in a completely self-contained way.

2. The case $n \leqslant 4$

The construction that proves the case $n \leqslant 4$ is actually quite natural, although the proof becomes somewhat technical at some points.

Theorem 1. *For $n \leqslant 4$, $\pi(\mathscr{U}_n) = n^2$.*

Proof. Since the $n \times n$ square can be divided into n^2 interior-disjoint unit squares, it is clear that $\pi(\mathscr{U}_n) \geqslant n^2$. Let us show that n^2 points suffice. We first show this for $n = 4$.

Let the vertices A, B, C, D of the square $ABCD$ have the coordinates $(0, 0)$, $(4, 0)$, $(4, 4)$ and $(0, 4)$, respectively. We choose 16 points at the following coordinates:

$$\left(1 - \varepsilon + i\frac{2 + 2\varepsilon}{3}, 1 - \varepsilon + j\frac{2 + 2\varepsilon}{3}\right), \ 0 \leqslant i, j \leqslant 3,$$

where ε is going to be chosen later. That way, the chosen 16 points represent a square lattice with the step $\frac{2+2\varepsilon}{3}$. Let $PQRS$ be the square that bounds this lattice (Fig. 1).

We need to show that each unit square inside the square $ABCD$ contains at least one of the chosen points in its interior (for a suitable ε). Let us first consider a unit square whose center is inside the square $PQRS$. Notice that, for each point inside the square $PQRS$, there exists at least one of the chosen 16 points at a distance from the observed point of no more than