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Enumerating graphs via even/odd dichotomy

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ABSTRACT

Following Hujdurović et al. (2016), an automorphism of a graph is said to be *even/odd* if it acts on the vertex set of the graph as an even/odd permutation. In this paper the formula for calculating the number of graphs of order n admitting odd automorphisms and the formula for calculating the number of graphs of order n without odd automorphisms are given together with their asymptotic estimates.

Such numbers are also considered for the subclass of vertex-transitive graphs. A positive integer n is a Cayley number if every vertex-transitive graph of order n is a Cayley graph. In analogy, a positive integer n is said to be a *vertex-transitive-odd number* (in short, a *VTO-number*) if every vertex-transitive graph of order n admits an odd automorphism. It is proved that there exists infinitely many VTO numbers which are square-free and have arbitrarily long prime factorizations. Further, it is proved that Cayley numbers congruent to 2 modulo 4, cubefree nilpotent Cayley numbers congruent to 3 modulo 4, and numbers of the form $2p$, p a prime, are VTO numbers. At the other extreme, it is proved that for a positive integer n the complete graph K_n and its complement are the only vertex-transitive graphs of order n admitting odd automorphisms if and only if n is a Fermat prime.

1. Introduction

The problem of determining the number of non-isomorphic graphs with n vertices was first considered by Redfield in 1927 [32], and somewhat latter by Polya [30]. He developed the famous Polya enumeration theorem with the use of which he was able to determine the number of graphs with a given number of vertices and edges. In this paper, similar techniques are employed to enumerate *even-closed* graphs, that is, graphs all of whose automorphisms are *even* — act as even permutations on the vertex set of the graph.

An automorphism of a graph is said to be *odd* if it acts on the vertex set of the graph as an odd permutation. Let E_n denote the number of non-isomorphic even-closed graphs of order n, and let O_n denote the number of non-isomorphic graphs of order n admitting an odd automorphism. In Section 3 exact formulas for the numbers E_n and O_n are given based on the Polya enumeration theorem (see Theorem 3.2 and Corollary 3.3). The asymptotic formulas for these numbers are given in Section 4 (see Proposition 4.1 and Corollary 4.2).

Almost all graphs have trivial automorphism groups, and consequently, almost all graphs are even-closed. In Section 5 the study of (non)existence of odd automorphisms in vertex-transitive graphs is pursued, arguably the most natural class of graphs to turn to for rich automorphism group. There are numbers n for which every vertex-transitive graph of order n, except for the complete graph and its complement, is even-closed. In Theorem 5.2 it is proved that this holds if and only if

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n is a Fermat prime. At the other extreme, there are numbers n for which every vertex-transitive graph of order n admits an odd automorphism; such is the case, quite surprisingly, with numbers of the form twice a prime (see Theorem 5.13).

In analogy to Cayley numbers, that is, numbers n for which every vertex-transitive graph of order n is in fact a Cayley graph [23], we will say that a number n is a *vertex-transitive-odd number* (in short, a *VTO number*) if every vertex-transitive graph of order n admits an odd automorphism. Theorem 1.1 stated below and proved in Section 5, summarizes the main results of this paper concerning VTO numbers. As a consequence of these results, in order to complete the classification of VTO numbers a detailed analysis, regarding existence of odd automorphisms in non-Cayley vertex-transitive graphs of order congruent to 2 modulo 4 and 3 modulo 4, is needed.

Theorem 1.1. *Let $n \geq 2$ be a positive integer. Then the following hold:*

(i) *If $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$ then n is a non-VTO number except for $n = 4$.*
(ii) *If $n \equiv 2 \pmod 4$ and n is a Cayley number then n is a VTO number.*
(iii) *If $n \equiv 2 \pmod 4$ and n is a non-Cayley number then n is a non-VTO number if and only if there exists an even-closed non-Cayley vertex-transitive graph of order n.*
(iv) *A multiple of an even non-VTO number is a non-VTO number.*
(v) *If $n \equiv 3 \pmod 4$ and n is a Cayley number then n is a VTO number if and only if it is cubefree nilpotent. There exist numbers satisfying these conditions that are a product of arbitrarily many distinct primes.*
(vi) *If $n \equiv 3 \pmod 4$ and n is a non-Cayley number then n is a non-VTO number if and only if either n is not cubefree or n is not nilpotent, or n is cubefree nilpotent and there exists an even-closed non-Cayley vertex-transitive graph of order n.*
(vii) *An odd multiple of a non-VTO number $n \equiv 3 \pmod 4$ is a non-VTO number.*

2. Preliminaries

Let Ω be a finite set, and let S_Ω denote the symmetric group on Ω, that is, the group of all permutations of the set Ω with the composition of functions as the group operation. When $|\Omega| = n$, we identify Ω with the set $\{1, \ldots, n\}$, and we write S_n instead of S_Ω. We denote with A_n the alternating group of degree n. For a permutation $g \in S_n$ and $k \in \{1, \ldots, n\}$ we denote by $c_k(g)$ the number of cycles of length k in the cycle decomposition of g. Then $\sum_{k=1}^{n} kc_k(g) = n$ and the vector $\mathbf{c}(\mathbf{g}) = (c_1(g), \ldots, c_n(g))$ can be associated with g. For $G \leq S_n$, we define the *cycle index* of G as the polynomial

$$Z(G) = \frac{1}{|G|} \sum_{g \in G} x_1^{c_1(g)} \cdot \ldots \cdot x_n^{c_n(g)}.$$

Since any two permutations $g_1, g_2 \in G \leq S_n$, with the same cycle decompositions contribute to the polynomial $Z(G)$ the same expression, it is sometimes convenient to express $Z(G)$ in terms of partitions of n. Let $h(\mathbf{c})$ denote the number of permutations of S_n with the cycle decomposition giving rise to a vector $\mathbf{c}$. Then

$$h(\mathbf{c}) = \frac{n!}{\prod k^{c_k} c_k!}, \tag{1}$$

and the cycle index of S_n can be written as

$$Z(S_n) = \frac{1}{n!} \sum_{\mathbf{c}} h(\mathbf{c}) \prod_k x_k^{c_k}, \tag{2}$$

where the sum is over all partitions/vectors $\mathbf{c}$ of n.

Let $q(x)$ be a polynomial and let $Z(G, q(x))$ be the polynomial obtained from $Z(G)$ by replacing each x_i with $q(x^i)$.

Let Ω and L be finite sets, and let $L^\Omega = \{f \mid f : \Omega \to L\}$. The set L^Ω can be thought of as the set of all colorings of the set Ω with colors from the set L. Let $G \leq S_\Omega$. For $f \in L^\Omega$ and $g \in G$, we define an induced action of G on the set L^Ω with $f^g = f \circ g$. Two functions $f_1, f_2 \in L^\Omega$ are called *G-equivalent*, if they belong to the same orbit under the induced action of G on L^Ω, or more precisely, if there exists $g \in G$ such that $f_1(g(\omega)) = f_2(\omega)$, for every $\omega \in \Omega$. Polya's enumeration theorem given below counts the number of orbits in this action.

Theorem 2.1 ([30]). *Let Ω and L be finite sets, and let $G \leq S_\Omega$. The number of orbits in the induced action of G on the set L^Ω is equal to $Z(G, |L|)$.*

Let us now count the number of non-isomorphic graphs of order n. Assume that the vertex set of a given graph X of order n is $\{1, \ldots, n\}$. Let $\Omega = \{\{i, j\} \mid 1 \leq i < j \leq n\}$, $L = \{0, 1\}$, and let $f_X : \Omega \to \{0, 1\}$ be defined by the rule

$$f_X(\{i, j\}) = 1 \text{ if and only if } \{i, j\} \in E(X).$$

Let F be the mapping from the set of all graphs of order n into the set $\{0, 1\}^\Omega$, defined by $F(X) = f_X$. It is not difficult to see that F is a bijective mapping. Moreover, it can be seen that the graphs X and Y are isomorphic if and only if the corresponding functions f_X and f_Y are S_n-equivalent. For $G \leq S_n$, we denote by $G^{(2)}$ the permutation group induced by the action of G on

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