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Applications of max-plus algebra to flow shop scheduling problems

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ABSTRACT

In this paper we consider applications of max-plus algebra to flow shop scheduling problems. Our aim is to show that max-plus algebra is useful for flow shop scheduling. We present two new solvable conditions in m-machine permutation flow shops using max-plus algebra. One of the conditions is found by considering a max-plus algebraic analogue of a proposition in linear algebra. The other is derived using a new framework, which associates a machine with a matrix and is the dual of the max-plus approach associating a job with a matrix by Bouquard, Lenté, and Billaut (2006). The framework is the first one which can deal with non-permutation flow shop problems based on max-plus algebra. Moreover, using the framework, we provide new simple proofs of some known results.

1. Introduction

Flow shop scheduling problems have been studied for several decades since Johnson's paper [22] on the two-machine flow shop problem.

The basic flow shop instance consists of m different machines, n jobs, and mn nonnegative values $p_{i,j}$ ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$), where $p_{i,j}$ specifies the time required by machine i for processing job j. Jobs flow from the first machine to the mth machine. Let $C_{i,j}$ be the completion time of job j at machine i.

We use the notation for scheduling problems suggested by Graham et al. [18]. A triplet $\alpha|\beta|\gamma$ describes a problem, where α denotes the machine environment, β provides details of constraints, and γ describes the objective to be minimized. α, β and γ considered in this paper are as follows. α is Fm, that denotes m-machine flow shops. The constraints in flow shops in the β field are:

- Permutation schedules (prmu). The same sequence of jobs is maintained between machines throughout (see e.g. [12,30]).
- Release dates (r_j). r_j is the earliest time at which job j can be processed (see e.g. [30]). Without r_j in the field, all jobs are available at time zero.

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- Starting times (s_i). s_i is the earliest time at which machine i can start processing (see e.g. [36,37]). Without s_i in the field, all machines are available at time zero.
- No-wait (no-wait). A job must be processed without waiting between two successive machines (see e.g. [20,30,32]). For simplicity, we consider only permutation schedules. It is known that the set of permutation schedules is not dominant when some jobs have zero processing times on certain machines (see e.g. [4]).
- No-idle (no-idle). A machine must process a job without idling, i.e., waiting for the next job (see e.g. [2,24])
- Blocking (blocking). A completed job at a machine has to remain on the machine until the downstream machine is available (see e.g. [20,30]). Here suppose the *First In First Out* discipline, then any blocking schedule is a permutation schedule.
- Busy (busy). A machine completing a job has to have the job until the next job comes (see e.g. [23]).

γ is the makespan ($C_{\max}$) or the total completion time ($\sum C_j$), where C_j denotes the completion time of job j at the last (mth) machine. However, the case where γ itself is specified means that the objective is arbitrary.

For example, given a job sequence $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$ for $Fm|\text{prmu}, r_j, s_i|\gamma$, the completion time $C_{i,j}$ can be computed through a set of recursive equations:

$$C_{i,\sigma(k)} = \max[C_{i-1,\sigma(k)}, C_{i,\sigma(k-1)}] + p_{i,\sigma(k)} \qquad (i = 2, \ldots, m;\ k = 2, \ldots, n) \tag{1a}$$
$$C_{1,\sigma(k)} = \max[r_{\sigma(k)}, C_{1,\sigma(k-1)}] + p_{1,\sigma(k)} \qquad (k = 2, \ldots, n) \tag{1b}$$
$$C_{i,\sigma(1)} = \max[s_i, C_{i-1,\sigma(1)}] + p_{i,\sigma(1)} \qquad (i = 2, \ldots, m) \tag{1c}$$
$$C_{1,\sigma(1)} = \max[s_1, r_{\sigma(1)}] + p_{1,\sigma(1)}. \tag{1d}$$

We can say that many theories of flow shop problems have been developed ad hoc. Burns and Rooker, Burns and Rooker [8,9], Szwarc [34] and Achugbue and Chin [1] considered only three-machine flow shops ($F3$). Though Nabeshima, Nabeshima [28,29] and Gupta [19] studied m-machine flow shops (Fm), the researches needed elaborate complicated computations. An exemption is the theoretical framework using critical paths by Szwarc [35] and Monma and Rinnooy Kan [27]. However, the framework can deal with only permutation schedules. Additionally, there are few papers concerning flow shop problems with the constraint r_j or s_i. A theoretical framework that describes flow shop problems in a unified manner and is useful to analyze the problems is demanded.

Max-plus algebra has been studied from the 1960s. A number of pioneering articles were published (see e.g. [10,13,15,31,33]). This algebra has found a lot of practical interpretations (see e.g. [3,11,14]).

In this paragraph, max-plus algebra will be introduced. Define $\mathbb{0} = -\infty$ and $\mathbb{1} = 0$, and denote the set $\mathbb{R} \cup \mathbb{0}$ by $\mathbb{R}_{\max}$. For elements $a, b \in \mathbb{R}_{\max}$, we define the operations $\oplus$ and $\odot$ by

$$a \oplus b = \max[a, b] \quad \text{and} \quad a \odot b = a + b.$$

The first operator,$\oplus$, is idempotent, commutative, associative and has a neutral element $\mathbb{0}$. The second operator, $\odot$, is commutative, associative, distributive on $\oplus$ and has a neutral element $\mathbb{1}$. Every element, except $\mathbb{0}$, is invertible: the inverse of x is denoted by x^{-1}. The two operators are extended to $m \times n$ matrices of elements of $\mathbb{R}_{\max}$. The element of a matrix $A \in \mathbb{R}_{\max}^{m\times n}$ in row i and column j is denoted by $(A)_{ij}$. The sum of matrices $A, B \in \mathbb{R}_{\max}^{m\times n}$ is defined as

$$(A \oplus B)_{ij} = (A)_{ij} \oplus (B)_{ij}$$

for all i, j. The product of $A \in \mathbb{R}_{\max}^{m\times l}$ and $B \in \mathbb{R}_{\max}^{l\times n}$ is defined as

$$(A \odot B)_{ij} = \bigoplus_{k=1}^{l} (A)_{ik} \odot (B)_{kj}$$

for all i, j. The standard orders, $\le$ and $\ge$, of real numbers are also extended to matrices (including vectors) componentwise, i.e., if A and B are of the same size then $A \le (\ge) B$ means that $(A)_{ij} \le (\ge)(B)_{ij}$ for all i, j. Henceforth, we in principle use not "max" and "+" but $\oplus$ and $\odot$, except for indices.

Our aim is to show that max-plus algebra is useful for flow shops. To our knowledge, there are only a few papers concerning the application of this algebra to flow shop problems. Giffler's paper [16] is the first application of max-plus algebra to scheduling problems. Lee [25] investigated cyclic job shops by using linear system theory on max-plus algebra. Cohen et al. [11] analyzed a flow-shop like production process and Hanen and Munier [21] modeled the cyclic scheduling problem using max-plus algebra. Max-plus algebra is applied to flow shop problems with minimal–maximal delays for two machines [6] and with minimal(–maximal) delays, setup and removal times [7,36,37].

In Section 2, we present a new solvable condition in m-machine permutation flow shops. The condition is a sufficient condition for Johnson's extended rule. We identify a link between the condition and a proposition in linear algebra. And we give a simple proof of the theorem that a no-wait flow shop makespan problem can be formulated as a Traveling Salesman Problem (TSP) using max-plus algebra. We present in Section 3 a new theoretical framework which associates a machine with a matrix and is the dual of the max-plus approach. Using the framework, we present a new solvable condition. Moreover, we show duality relationships between different flow shops. Finally, we provide a concluding remark in Section 4.

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