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Note

On some open problems concerning quorum colorings of graphs

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ABSTRACT

A partition $\pi = \{V_1, V_2, \dots, V_k\}$ of the vertex set V of a graph G into k color classes V_i, with $i \in \{1, \dots, k\}$, is called a quorum coloring if, for every vertex $v \in V$, at least half of the vertices in the closed neighborhood $N[v]$ of v have the same color as v. The maximum order of a quorum coloring of G is called the quorum coloring number of G and is denoted $\psi_q(G)$. In this paper, we give answers to three open problems stated in 2013 by Hedetniemi, Hedetniemi, Laskar and Mulder. In particular, we show that the decision problem associated with $\psi_q(G)$ is NP-complete, and we prove that for any graph G on n vertices, $\psi_q(G) + \psi_q(\overline{G}) \leq n + 2$, where $\overline{G}$ is the complement of G.

1. Introduction

Let $G = (V, E)$ be a simple graph of order $n = |V|$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v in G is $d_G(v) = |N(v)|$. A vertex of G with degree one is a *leaf* of G. The *maximum* and *minimum vertex degrees* in G are denoted by $\Delta(G)$ and $\delta(G)$, respectively.

A partition $\pi = \{V_1, V_2, \dots, V_k\}$ of the vertex set V of a graph G into k color classes V_i, with $i \in \{1, \dots, k\}$, is called a *quorum coloring* if, for every vertex $v \in V$, at least half of the vertices in the closed neighborhood $N[v]$ have the same color as v. The maximum order of a quorum coloring of G is called the *quorum coloring number* of G and is denoted $\psi_q(G)$. A quorum coloring of order $\psi_q(G)$ is called a ψ_q-*coloring*. Quorum colorings were introduced by Hedetniemi, Hedetniemi, Laskar and Mulder [6]. The concept of quorum coloring is closely related to the concept of defensive alliances in graphs introduced by Kristiansen, Hedetniemi and Hedetniemi [5]. Indeed, a *defensive alliance* in G is a subset S of V such that, for every vertex $v \in S$, we have $|N[v] \cap S| \geq |N(v) \cap (V \setminus S)|$. Hence every color class of a ψ_q-coloring is a defensive alliance. Note that Haynes and Lachniet in [4] were the first to introduce the problem of partitioning the vertex set V into defensive alliances. This problem was also studied earlier by Eroh and Gera [1]. However, we will adopt in this paper the definitions and notations given in [6].

The corona $G \circ K_1$ of a graph G is the graph made from G by appending a vertex of degree one to each vertex of G. Let $\overline{G}$ denote the complement of the graph G.

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In [6], Hedetniemi et al. raised the following problems.

1. What is the complexity of the following decision problem:
   QUORUM-K
   Instance: Graph $G=(V,E)$, positive integer $K \le |V|$.
   Question: Does G have a quorum coloring of order at least K?
2. What are good Gaddum–Nordhaus bounds for $\psi_q(G)+\psi_q(\overline{G})$ and $\psi_q(G)\times\psi_q(\overline{G})$?

Moreover, the authors [6] posed the following conjecture.

Conjecture 1. *If G is a graph of order $n \ge 4$, then $4 \le \psi_q(G)+\psi_q(\overline{G}) \le n+2$.*

3. It is easy to see that for any graph G, $\psi_q(G) \le \psi_q(G\circ K_1)$. But is a more refined result possible? For example, when is this inequality strict?

In this paper, we first show that problem QUORUM-K is NP-complete. Then we prove the right inequality of Conjecture 1 and we answer Question 3.

2. Preliminary results

We begin by recalling some important results that will be useful in our investigations. Obviously, $1 \le \psi_q(G) \le n$ for every graph G of order n. A characterization of the graphs G of order n with $\psi_q(G)=n$ is given in [6] as follows.

Theorem 2 (*Hedetniemi et al. [6]*)**.** *If G is a graph of order n, then $\psi_a(G)=n$ if and only if $\Delta(G)\le 1$.*

In contrast to the upper bound, the characterization of graphs G with $\psi_q(G)=1$ remains open. The following result shows that such graphs have minimum degree at least two.

Proposition 3 (*Hedetniemi et al. [6]*)**.** *If G is a graph with order $n\ge 2$ and minimum degree $\delta\in\{0,1\}$, then $\psi_q(G)\ge 2$.*

Restricted to graphs G with minimum degree at least two, the authors [6] improved the upper bound on the quorum coloring number.

Proposition 4 (*Hedetniemi et al. [6]*)**.** *If G is a graph with order n and minimum degree $\delta\ge 2$, then $\psi_q(G)\le\lfloor n/2\rfloor$.*

The next two results are due to Eroh and Gera.

Proposition 5 (*Eroh and Gera [1]*)**.** *For each pair k and n of positive integers with $k\le n$, there exists a graph G of order n and $\psi_q(G)=k$, except for $k=1$ and $n\in\{2,4\}$.*

Theorem 6 (*Eroh and Gera. [1]*)**.** *Let G be a graph with minimum degree δ. Then $\psi_q(G)\le\left\lfloor \frac{n}{\lceil\frac{\delta+1}{2}\rceil}\right\rfloor$.*

We close this section by giving an upper bound for the quorum coloring number of any graph G in terms of the order, and the maximum and minimum degrees.

Proposition 7. *For any graph G, $\psi_q(G)\le 1+\left\lfloor\frac{n-\lceil\frac{\Delta(G)+1}{2}\rceil}{\lceil\frac{\delta(G)+1}{2}\rceil}\right\rfloor$.*

Proof. Let π be a ψ_q-coloring of G and v a vertex of maximum degree. Assume that v belongs to a color class A of π. Clearly $|A|\ge\lceil\frac{\Delta(G)+1}{2}\rceil$ and so $|V\setminus A|\le n-\lceil\frac{\Delta(G)+1}{2}\rceil$. Since every color class of π contains at least $\lceil\frac{\delta(G)+1}{2}\rceil$ vertices, we obtain that the vertices of $V\setminus A$ are contained in at most $\left\lfloor\frac{|V\setminus A|}{\lceil\frac{\delta(G)+1}{2}\rceil}\right\rfloor$ distinct color classes. Therefore $\psi_q(G)\le 1+\left\lfloor\frac{|V\setminus A|}{\lceil\frac{\delta(G)+1}{2}\rceil}\right\rfloor\le 1+\left\lfloor\frac{n-\lceil\frac{\Delta(G)+1}{2}\rceil}{\lceil\frac{\delta(G)+1}{2}\rceil}\right\rfloor$. □

As an immediate consequence to Proposition 7, we have the following.

Corollary 8. *For any graph G, $\psi_q(G)\le 1+n-\lceil\frac{\Delta(G)+1}{2}\rceil$.*