Discrete Applied Mathematics 247 (2018) 327–340

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

The packing number of the double vertex graph of the path graph

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ARTICLE INFO

Article history:
Received 16 November 2017
Received in revised form 23 March 2018
Accepted 28 March 2018
Available online 12 July 2018

Keywords:
Double vertex graph
Packing number
Error correcting codes
Binary codes

ABSTRACT

Neil Sloane showed that the problem of determining the maximum size of a binary code of constant weight 2 that can correct a single adjacent transposition is equivalent to finding the packing number of a certain graph. In this paper we solve this open problem by finding the packing number of the double vertex graph of the path graph. This double vertex graph is isomorphic to Sloane's graph. Our solution implies a conjecture of Rob Pratt about the ordinary generating function of sequence A085680.

1. Introduction

Let G be a graph of order $n \geq 3$. The *double vertex graph* $F_2(G)$ of G is the graph whose vertex set consists of all 2−subsets of $V(G)$, where two vertices are adjacent in $F_2(G)$ whenever their symmetric difference is an edge of G. Such graphs were introduced and studied by Alavi, Behzad and Simpson from the point of view of the planarity [2]. Later, T. Rudolph [4] redefined the double vertex graphs, with the name of *symmetric squares of graphs*, in connection with quantum mechanics and the graph isomorphism problem. Rudolph presented examples of cospectral non-isomorphic graphs G and H, such that the corresponding double vertex graphs are non-cospectral.

The concept of double vertex graph of a graph can be extended in a natural way if we consider the k-subsets of $V(G)$, with $k \in \{1, \dots, n-1\}$, instead of the 2-subsets. Such a graph is often denoted by $F_k(G)$, and has been redefined, independently, by several authors [9,16,22]. Following [9], we will refer to $F_k(G)$ as the *k-token graphs* of G. The k-token graph of a graph has been extensively studied, see for instance [1,3,6–8,11–15,21].

We recall that a set $S \subseteq V(G)$ is a *packing set* of G if every pair of distinct vertices $u, v \in S$ satisfy $d_G(u, v) \geq 3$, where $d_G(u, v)$ denotes the distance between u and v in G. The *packing number* $\rho(G)$ of G is the maximum cardinality of a packing set of G. Note that the notion of packing set can be extended to infinite graphs in a natural way. In this paper we are interested in determining the packing number $\rho(F_2(P_n))$ of the double vertex graph of P_n the path graph of order n. As we shall see in the next section, the sequence produced by $\rho(F_2(P_n))$ corresponds to the sequence A085680 in OEIS [20].

1.1. Packing number and error correcting codes

Let n be a positive integer and let $w \in \{0, 1, \dots, n\}$. A *binary code of length n and weight w* is a subset S of $\mathbb{F}_2^n$ such that every element in S has exactly w 1's and $n - w$ 0's. The elements of S are called *codewords*. For a binary vector $u \in \mathbb{F}_2^n$, let

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https://doi.org/10.1016/j.dam.2018.03.085

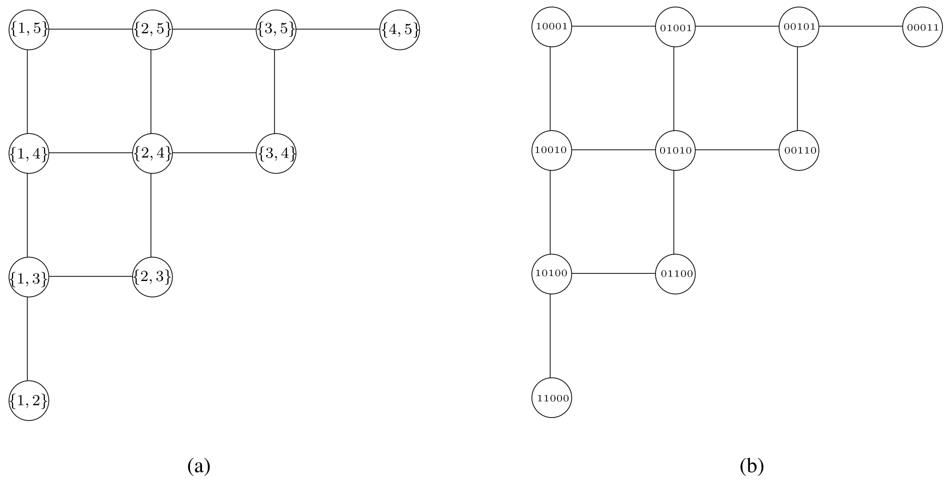

Fig. 1. (a) The double vertex graph of P_5. (b) Graph Γ_5.

$B_e(u)$ denote the set of all binary vectors in $\mathbb{F}_2^n$ that can be obtained from u as a consequence of certain error e. For example, e can be the deletion or the transposition of bits.

A subset C of $\mathbb{F}_2^n$ is said to be an *e-deletion-correcting code* if $B_e(u) \cap B_e(v) = \emptyset$, for all $u, v \in C$ with $u \neq v$. A classical problem in coding theory is to find the largest e-deletion-correcting code. In the survey of Sloane [18] we can see results about single-deletion-correcting-codes. In Butenko et al. [5] and in the web page [17] we can see results about transposition error correcting codes.

The integer sequence produced by the size of largest binary code of length n and constant weight 2 that can correct a single adjacent transposition, has been studied since 2003 and corresponds to the sequence A085680 in OEIS [20]. The range of values of n for which A085680(n) is known is from 2 to 50.

The problem of determining A085680(n) was formulated in the setting of graph theory by Sloane [19] as follows: let Γ_n be the graph whose vertices are all binary vectors of length n and constant weight 2, and two vertices are adjacent if and only if one can be obtained from the other by transposing a pair of adjacent bits. From the definition of Γ_n and the requirement that $B_e(u) \cap B_e(v) = \emptyset$, for any distinct vertices u and v of Γ_n, it follows that any binary code C with constant weight 2 can correct a single adjacent transposition if and only if C is a packing set of Γ_n. Therefore, the maximum cardinality of such a code C is equal to the packing number $\rho(\Gamma_n)$ of Γ_n. In other words, $A085680(n) = \rho(\Gamma_n)$.

Surprisingly, the graph Γ_n is isomorphic to $F_2(P_n)$. We end this subsection by showing the affirmation that $\Gamma_n \simeq F_2(P_n)$. Suppose that $V(P_n) = \{1, \ldots, n\}$ and that $E(P_n) = \{\{i, i+1\} : 1 \leq i \leq n-1\}$. Consider $f : V(F_2(P_n)) \to V(\Gamma_n)$ defined as follows: for $A \in V(F_2(P_n))$, let $f(A) := (a_1, \ldots, a_n)$, where $a_i = 1$ if $i \in A$ and $a_i = 0$, otherwise. It is easy to check that f is an isomorphism. Hence A085680(n) $= \rho(F_2(P_n))$. In Fig. 1 we show the double vertex graph of P_5 and the graph Γ_5.

1.2. Main result

The exact value for A085680(n) before march of 2017 was known only for $n \in \{2, \ldots, 25\}$. Later, Rob Pratt reported in [20] the exact values of A085680(n) for $n \in \{26, \ldots, 50\}$, and posed the following conjecture about the ordinary generating function of this sequence.

Conjecture 1. $\sum_{n\geq 0} A085680(n+2)x^n = \frac{1-x+x^2-x^{10}+x^{11}}{(1-x)^2(1-x^5)}$.

Consider the following sequence:

$$a(n) := \begin{cases} \dfrac{1}{10}\left(n^2+n+20\right) & \text{if } n \equiv 0 \pmod 5 \text{ or } n \equiv 4 \pmod 5, \\ \dfrac{1}{10}\left(n^2+n+18\right) & \text{if } n \equiv 1 \pmod 5 \text{ or } n \equiv 3 \pmod 5, \\ \dfrac{1}{10}\left(n^2+n+14\right) & \text{if } n \equiv 2 \pmod 5. \end{cases}$$

Unless otherwise stated, for the rest of the paper, $a(n)$ is as above. The following remark follows immediately from the definition of $a(n)$.