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Note

On the number of maximal independent sets in minimum colorings of split graphs

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ABSTRACT

The largest number of color classes that are dominating sets (or, equivalently, maximal independent sets) among all $\chi(G)$-colorings of a graph G is called the dominating-χ-color number of G, and denoted $d_\chi(G)$. It was shown in Arumugam et al. (2011) that determining whether $d_\chi(G) \geq 2$ is NP-complete even in 3-chromatic graphs G. In this note, we prove that in split graphs the dominating-χ-color number equals to 1 if and only if its domination number is greater than 2. While such graphs can be efficiently recognized, we prove that for split graphs with domination number at most 2 the decision version of the problem of determining dominating-χ-color number is NP-complete. We also present existence results for split graphs attaining prescribed values of dominating-χ-color numbers and some other relevant parameters.

1. Introduction

Given a graph G and a positive integer k, a function $c : V(G) \to [k]$ such that $c(u) \neq c(v)$ if $uv \in E(G)$, is called a *proper coloring* of G. If such a function exists, G is called *k-colorable*, while elements in the image of c are *colors*. Note that a proper coloring c can be uniquely presented by the ordered partition of $V(G)$ into color classes, which we write as $\mathcal{C} = (C_1, \ldots, C_k)$, and, by abuse of language, we call both the function c and the partition $\mathcal{C}$ a *k-coloring* of G.

The minimum k for which G is k-colorable is the *chromatic number* of G, and is denoted by $\chi(G)$. By *χ-coloring of G* we mean its $\chi(G)$-coloring, i.e., a coloring with minimum number of colors. The following (trivial) observation is essential for defining the main concept of this paper, i.e., the dominating-χ-coloring.

Observation 1 ([13]). *Every graph G contains a χ-coloring with the property that at least one color class is a dominating set in G.*

Recall that *a dominating set* of a graph G is a set $D \subseteq V(G)$ such that every vertex not in D is adjacent to at least one vertex from D. We say that a vertex u *dominates* a vertex v, if they are adjacent, while a set $X \subset V(G)$ *dominates* a set $Y \subset V(G)$, if every vertex in $Y \setminus X$ is adjacent to a vertex in X. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. Also, recall the notation $N_G(x) = \{y \in V(G) \mid xy \in E(G)\}$, where x is a vertex in a graph G, and $N_G(X) = \cup_{x\in X} N(x)$, where X is a set of vertices in G. (When the graph G is understood from the context, we may also write simply $N(x)$, resp. $N(X)$.) We say that $X, X \subset V(G)$, *totally dominates* a set Y if $Y \subseteq N(X)$. A vertex $x \in V(G)$ is a *universal vertex* in G if $N(x) = V(G) \setminus \{x\}$.

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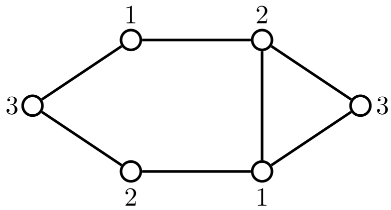

Fig. 1. Graph G with $d_\chi(G) = \chi(G)$ with a 3-coloring such that all color classes are dominating.

Motivated by Observation 1, Arumugam et al. [3] defined the dominating-χ-color number as follows. Among all χ-colorings of G, a coloring with the maximum number of color classes that are dominating sets of G is called a *dominating-χ-coloring* of G. The number of color classes that are dominating sets in a dominating-χ-coloring of G is defined to be the *dominating-χ-color number* of G, denoted by $d_\chi(G)$. Hence for a $\chi(G)$-coloring $\mathcal{C}$ of G, if $d_\mathcal{C}$ denotes the number of color classes in $\mathcal{C}$ that are also dominating sets of G, then $d_\chi(G) = \max\{d_\mathcal{C}\}$, where the maximum is taken over all χ-colorings of G. A practical application for dominating-χ-coloring arising from two hierarchical objectives for scheduling processors in a parallel processor system was presented in [2].

The following lower bound and upper bound on $d_\chi(G)$ for an arbitrary graph G are immediate consequences of definitions:

$$1 \le d_\chi(G) \le \chi(G). \tag{1}$$

The lower bound in (1) is achieved in many classes of graphs (e.g., see Theorem 3). As observed also in [3] the upper bound in (1) is attained in uniquely χ-colorable graphs G. In Fig. 1, graph G with $\chi(G) = 3$ is depicted, which is not uniquely 3-colorable but contains a 3-coloring in which all color classes are dominating.

Given a graph G with $\chi(G) = k$ and an integer r such that $1 \le r \le k$ it is NP-complete to determine whether $d_\chi(G) \ge r$. Moreover, as shown in [2], the problem stays NP-complete even in the case $k = 3, r = 2$. It is easy to see that if G is a bipartite graph with no isolated vertices, then $d_\chi(G) = 2$ [3]. Apart from this trivial case, the computational complexity of this problem has not been established in other classes of graphs.

A graph G is a *split graph* if $V(G)$ can be partitioned into sets K and I, where K induces a complete graph, and I is an independent set. Although they form a relatively simple class of graphs, split graphs were studied in a number of papers. In particular, algorithmic behavior of many graph parameters was investigated in split graphs, cf. some recent papers [1,4,5]. Split graphs also have several applications and contain interesting graph classes such as threshold graphs (see [11] and the references therein).

The main result of this note is dichotomy of the dominating-χ-coloring problem in the class of split graphs. We first prove that a connected split graph G has $d_\chi(G) = 1$ if and only if G has domination number greater than 2. This implies the existence of a polynomial time algorithm for deciding whether $d_\chi(G) = 1$ holds for a split graph G; on the other hand, we prove that for a split graph G with $\gamma(G) \le 2$, and an integer ℓ, $\ell \le \chi(G)$, it is NP-complete to determine whether $d_\chi(G) \ge \ell$. (Similar kind of dichotomies were established for many other graph parameters.) We then also prove two realization results with respect to split graphs. One of them states that for arbitrary integers $m \ge 1, n \ge 1$, where $m \le n+1$, there exists a split graph G with $d_\chi(G) = m$ and $\delta(G) = n$. In the next section we prove these results, while in the concluding section we give some additional observations, and pose some problems on other classes of chordal graphs.

2. Split graphs

We start by the following lemma that will help in characterizing the split graphs G with $d_\chi(G) = 1$.

Lemma 2. *If G is a graph such that for any maximal independent set S there exists no independent set I that totally dominates S, then $d_\chi(G) = 1$.*

Proof. Let G be a graph such that any maximal independent set S in G is not totally dominated by an independent set I. Let $\mathcal{C} = (C_1, \ldots, C_{\chi(G)})$ be a $\chi(G)$-coloring of G, containing a color class, which is a dominating set of G. We may assume without loss of generality that C_1 is a dominating set. Since C_1 is a dominating and an independent set, it is a maximal independent set. By the statement's assumption, there is no independent set I that totally dominates C_1. Note that such an independent set I, in order to totally dominate C_1, would necessarily need to be disjoint with C_1. Hence we derive that there is no independent set in $G - C_1$ that totally dominates C_1. In particular, no color class C_i (totally) dominates C_1, which implies that no color class C_i, $i \ne 1$, dominates $V(G)$. Thus, $d_\chi(G) = 1$. $\square$

Theorem 3. *If G is a connected split graph, then $d_\chi(G) = 1$ if and only if $\gamma(G) > 2$.*

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