Systems & Control Letters 120 (2018) 1–8

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Singular linear systems on Lie groups; equivalence

Victor Ayala [a,*], Philippe Jouan [b]

[a] Instituto de Alta Investigación, Universidad de Tarapacá, Sede Iquique Chile and Universidad Católica del Norte, Antofagasta, Chile
[b] Lab. R. Salem, CNRS UMR 6085, Université de Rouen, avenue de l'université BP 12, 76801 Saint-Étienne-du-Rouvray, France

ARTICLE INFO

Article history:
Received 1 December 2017
Received in revised form 17 April 2018
Accepted 16 July 2018

Keywords:
Singular systems
Control systems
Lie groups
Homogeneous spaces

ABSTRACT

In this paper, we define different kinds of singular systems on Lie groups and we analyze some of them. Furthermore, we state sufficient conditions for a general nonlinear singular system defined on a manifold to be equivalent by diffeomorphism to one of these models. Some examples are computed.

1. Introduction

The purpose of this paper is to design models of nonlinear singular systems by means of the so-called linear systems on Lie groups, which are actually nonlinear, despite their name that comes from their similarity with linear systems in $\mathbb{R}^n$.

More accurately a control-affine system on a Lie group is said to be linear if its drift is an infinitesimal automorphism (see Bourbaki [1]), called linear vector field in a geometric control context, and the controlled fields are right (or left)-invariant. Thanks to the Equivalence Theorem of [2] these systems appear as models for a large class of nonlinear systems: the finite ones, that is the systems whose generated Lie algebra is finite dimensional. To be more precise the Equivalence Theorem states that (under some technical assumptions) a finite system is equivalent by diffeomorphism to a linear system on a Lie group or a homogeneous space.

It is worthwhile to notice that in order to state the Equivalence Theorem in its full generality it is necessary to extend the definition of linear systems to homogeneous spaces.

On the other hand linear and nonlinear singular equations and control systems have attracted a lot of interest under different names: differential–algebraic equations (DAE) or systems, descriptors, degenerate systems. Good accounts of the linear theory can be found in the books [3,4]. The more recent book [5] deals also with nonlinear DAE, analyzed through linearization (for this approach see also [6,7]) and contains a detailed bibliography. Our approach is more geometric (see for instance [8–11]) but opposite to what happens for general nonlinear differential–algebraic systems defined on manifolds, the Lie structure allows us to get a natural splitting into the differential and algebraic parts, which is moreover global. The counterpart is that we deal with systems that generate a finite dimensional Lie algebra only.

Our goal to design models of nonlinear singular systems is reached in three steps:

1. The first one consists in defining models of singular linear systems on a Lie group G, that is systems of the kind
$$E_g.\dot{g} = \mathcal{X}_g + Y_g + \sum_{j=1}^{m} u_j Y_g^j$$
where $\mathcal{X}$ is a linear vector field, Y and the Y^js are right-invariant, and E_g is a noninvertible linear map defined on each tangent space T_gG. They were introduced in [12] in the case where E is a derivation of the Lie algebra $\mathfrak{g}$ of G and $E_g = TR_g.E.TR_{g^{-1}}$.
2. In a second step we should analyze these models. This analysis is not essential to state an equivalence theorem but it validates the interest of the models. They would be worthless if their analysis was not possible.
3. To finish we have to state and prove the equivalence of finite singular systems with our models after having checked that can be extended to homogeneous spaces.

The first step is realized in Section 3, after having recalled the basic definitions in Section 2. It is natural to require that the linear mapping E_g that acts on the tangent space T_gG be related to the Lie structure. This leads to $E_g = TR_g.E.TR_{g^{-1}}$ where E is a derivation or a noninvertible morphism of the Lie algebra $\mathfrak{g}$ of G. Another possibility is to define a singular linear system using a

* Corresponding author.
E-mail addresses: vayala@uta.cl (V. Ayala), philippe.jouan@univ-rouen.fr (P. Jouan).

https://doi.org/10.1016/j.sysconle.2018.07.010
0167-6911/

noninvertible Lie group morphism θ, that is to consider systems of the kind

$$T_g\theta.\dot{g} = \mathcal{X}_{\theta(g)} + Y_{\theta(g)} + \sum_{j=1}^{m} u_j Y^j_{\theta(g)}.$$

These systems are quite different from the previous ones, in particular the usual existence and uniqueness of the solutions does not hold.

Section 4 is devoted to the analysis of singular systems defined by Lie algebra morphisms. It is shown that under some additional (but natural) assumptions these systems can be analyzed, that is decomposed into a nonsingular part and an algebraic one. As well the additional assumptions than the analysis itself are close to the usual ones for singular linear systems in $\mathbb{R}^n$. The analysis of singular systems defined by a derivation has been previously made in [13] and the one of singular systems defined by Lie group morphisms is postponed to Section 7.

The third step is done in Section 6 where the equivalence theorem is stated. Its proof, that makes use of the Equivalence Theorem of [2] recalled in the Appendix, follows the technical Section 5 where it is shown that the models defined by derivation or morphism can be extended to homogeneous spaces.

Section 7 deals with the group morphism case. As explained above the reasons to deal with these systems in a different part are that on the one hand their analysis is very easy but on the other one the proof of the equivalence Theorem is quite different for them. They have also the drawback that neither the existence nor the uniqueness of the solutions are guaranteed.

Two examples are presented in Section 8. The first one is a decomposition into the horizontal and the vertical part of a morphism model on the 2-dimensional affine group. In the second example an algebraic system on the Heisenberg group is solved.

2. Basic definitions and notations

In this section the definition of linear vector fields and some of their properties are recalled. More details can be found in [2] (see also [14]).

Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra (the set of right-invariant vector fields, identified with the tangent space at the identity).

The right (resp. left) translation by $g \in G$ is denoted by R_g (resp. L_g) and its differential at the point h by T_hR_g, or by TR_g if no confusion can happen (resp. T_hL_g or TL_g).

A vector field on G is said to be *linear if its flow is a one parameter group of automorphisms*. Notice that a linear vector field is consequently analytic and complete.

The following characterization will be useful in the sequel.

Characterization of linear vector fields

Let $\mathcal{X}$ be a vector field on a connected Lie group G. The following conditions are equivalent:

1. *$\mathcal{X}$ is linear;*
2. *$\mathcal{X}$ belongs to the normalizer of $\mathfrak{g}$ in the algebra $V^\omega(G)$ of analytic vector fields of G, that is*
$$\forall Y \in \mathfrak{g} \qquad [\mathcal{X}, Y] \in \mathfrak{g}, \tag{1}$$
and verifies $\mathcal{X}(e) = 0$;
3. *$\mathcal{X}$ verifies*
$$\forall g, g' \in G \qquad \mathcal{X}_{gg'} = TL_g.\mathcal{X}_{g'} + TR_{g'}.\mathcal{X}_g \tag{2}$$

According to (1) one can associate to a given linear vector field $\mathcal{X}$ the derivation D of $\mathfrak{g}$ defined by:

$$\forall Y \in \mathfrak{g} \qquad DY = -[\mathcal{X}, Y],$$

that is $D = -\text{ad}(\mathcal{X})$. The minus sign in this definition comes from the formula $[Ax, b] = -Ab$ in $\mathbb{R}^n$. It also enables to avoid a minus sign in the formula:

$$\forall Y \in \mathfrak{g}, \quad \forall t \in \mathbb{R} \qquad \varphi_t(\exp Y) = \exp(e^{tD}Y),$$

where $(\varphi_t)_{t\in\mathbb{R}}$ stands for the flow of $\mathcal{X}$.

An *affine vector field* is an element of the normalizer $\mathfrak{N}$ of $\mathfrak{g}$ in $V^\omega(G)$, that is

$$\mathfrak{N} = \text{norm}_{V^\omega(G)}\mathfrak{g} = \{L \in V^\omega(G);\ \forall Y \in \mathfrak{g}, \quad [L, Y] \in \mathfrak{g}\},$$

so that an affine vector field is linear if and only if it vanishes at the identity.

It can be shown (see [2]) that an affine vector field can be uniquely decomposed into a sum $L = \mathcal{X} + Y$ where $\mathcal{X}$ is linear and Y right-invariant.

Let $\mathcal{X}$ be a linear vector field and F its translation to the tangent space at the identity, that is $F_g = TR_{g^{-1}}.\mathcal{X}_g$ for all $g \in G$. The following formulas are computed in [15].

1. The differential of F at the point g is:
$$T_gF = (D + \text{ad}(F_g)) \circ TR_{g^{-1}}. \tag{3}$$
2. For $g = \exp(tY)$ one has
$$F(\exp tY) = \sum_{k=1}^{+\infty}(-1)^{k-1}\frac{t^k}{k!}\text{ad}^{k-1}(Y)DY. \tag{4}$$

Let us consider now the nonlinear control system on a manifold M:

$$\dot{x} = \frac{d}{dt}x = f(x) + \sum_{j=1}^{m} u_j g_j(x),$$

where f and the g_j's are smooth vector fields, and $u = (u_1, \ldots, u_m)$ belongs to $\mathbb{R}^m$. Let $\mathcal{L}$ be the Lie algebra of vector fields generated by f and the g_js. The **rank of the system** at a point x is the dimension of $\{\xi(x);\ \ \xi \in \mathcal{L}\}$. The system satisfies the **rank condition** if this rank is equal to the dimension of M at all points, that is if $\{\xi(x);\ \xi \in \mathcal{L}\} = T_xM$ for all $x \in M$.

The **admissible inputs** are the locally essentially bounded functions from $[0, +\infty[$ to $\mathbb{R}^m$.

3. Different types of singular systems on Lie groups

Let us consider a linear system on a Lie group G. It has the following form:

$$\dot{g} = \mathcal{X}_g + Y_g + \sum_{j=1}^{m} u_j Y^j_g, \tag{5}$$

where $\mathcal{X}$ is a linear field on G and Y and the Y^j's are right-invariant. The drift vector field is here the affine vector field $\mathcal{X} + Y$ and the system is right-invariant if $\mathcal{X} = 0$.

This system becomes a singular one if some noninvertible mapping is applied to $\dot{g}$. In the paper [12] the authors consider a derivation E of the Lie algebra $\mathfrak{g}$ of G (identified with the tangent space T_eG at the identity) and define the singular system as

$$E_g.\dot{g} = \mathcal{X}_g + Y_g + \sum_{j=1}^{m} u_j Y^j_g, \tag{6}$$

where $E_g = TR_g \circ E \circ TR_{g^{-1}}$.

Another possibility is to replace the derivation E by a noninvertible Lie algebra morphism P, that is to consider the model

$$P_g.\dot{g} = \mathcal{X}_g + Y_g + \sum_{j=1}^{m} u_j Y^j_g, \tag{7}$$