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Stabilization of nonlinear time-delay systems: Distributed-delay dependent impulsive control ☆

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A R T I C L E I N F O

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A B S T R A C T

This technical note studies impulsive stabilization of general nonlinear systems with time-delay. Distributed time-delay is considered in the proposed nonlinear impulsive controller. Using Lyapunov–Razumikhin method, an exponential stability criterion is constructed, which is then applied to investigate stabilization of a linear time-delay system under linear distributed-delay dependent impulsive control. Sufficient conditions on the system parameters, impulsive control gains, impulsive instants and distributed delays are obtained in the form of an inequality for global exponential stability. In these results, it is shown that an unstable time-delay system can be successfully stabilized by distributed-delay dependent impulses. It is worth noting that the proposed impulsive controllers are independent of the system states at each impulsive instant, and the states with distributed delays play the key role in the stabilization process. A numerical example is provided to demonstrate the efficiency of the main results.

1. Introduction

Time-delay systems have been intensively studied in the past decades, mainly due to the ubiquity of time delays in physical processes such as proliferation process for solid avascular tumor [1], scattering process [2], milling process [3], and temperature control [4]. Stability is one of the fundamental issues in system design, analysis and control. Recently, impulsive control has been shown to be a powerful approach to stabilize time-delay systems, and various stability and stabilization results have been obtained for impulsive time-delay systems (see [5–7]).

In general, it takes non-neglectable time to sample, process and transfer the impulsive information in the controller. Therefore, it is practically needed to consider time-delay in impulsive controllers. There are many recent attempts to investigate time-delay systems with delayed impulses (see, for example, [8–17]). Typically, [15] studied stabilization of a class of delay-free nonlinear systems by linear delayed impulses, and then [16] investigated the exponential stability of time-delay systems with nonlinear delayed impulses. The most recent results about delay-dependent impulsive control of time-delay systems were reported in [17]. The author studied a class of linear systems with both discrete and distributed delays subject to delayed impulses. However, from the control point of view, the delay part of the impulsive controller in [17] may not contribute to the system stability, which could be contrary to what the authors have claimed in [17]. See Remark 1 for detailed discussions.

On the other hand, in the above mentioned references, only discrete delays were considered in the impulsive controllers. As another type of time-delay, distributed delay has been widely employed in biological and industrial systems to describe time-delay in the spread of disease [18], network connections [19], epidemic model [20], etc. To our best knowledge, distributed-delay dependent impulsive control has not been studied for stabilization of time-delay systems. The idea of distributed-delay dependent impulsive control is as follows: the jumps of systems states do not rely on the states at each impulsive instant or the states at history time, but depend on the accumulation (or average) of the system states over a history time period.

Motivated by the above discussion, in this technical note, we study stabilization problem of general nonlinear time-delay systems by distributed-delay dependent impulsive control. Stability criteria for the impulsive control systems are constructed by using Razumikhin technique and Lyapunov functions. The remainder of this technical note is organized as follows. In Section 2, stabilization problem of time-delay systems is formulated and a class of distributed-delay dependent impulsive controllers is proposed. In Section 3, we construct sufficient conditions for impulsive stabilization of nonlinear and linear time-delay systems, respectively. An example with several numerical simulations are provided in Section 4 to illustrate the main results. Finally, we summarize our results in Section 5.

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2. Preliminaries

Through out this technical note, let $\mathbb{N}$ denote the set of positive integers, $\mathbb{R}$ the set of real numbers, $\mathbb{R}^+$ the set of nonnegative real numbers, and $\mathbb{R}^n$ the n-dimensional real space equipped with the Euclidean norm $\|\cdot\|$. For any matrix $A \in \mathbb{R}^{n\times n}$, let A^T denote the transpose of A, $\lambda_{max}(A)$ the largest eigenvalue of A, and $\|A\| = \sqrt{\lambda_{max}(A^TA)}$, i.e., the norm of A induced by the Euclidean norm. Denote $I \in \mathbb{R}^{n\times n}$ the $n\times n$ identity matrix. For $a, b \in \mathbb{R}$ with $a < b$ and $S \subseteq \mathbb{R}^n$, we define $\mathcal{PC}([a,b],S) = \Big\{\psi : [a,b] \to S \Big| \psi(t) = \psi(t^+)$, for any $t \in [a,b)$; $\psi(t^-)$ exists in S, for any $t \in (a,b]$; $\psi(t^-) = \psi(t)$ for all but at most a finite number of points $t \in (a,b]\Big\}$, where $\psi(t^+)$ and $\psi(t^-)$ denote the right and left limits of function ψ at t, respectively. For a given constant $\tau > 0$, the linear space $\mathcal{PC}([-\tau,0],\mathbb{R}^n)$ is equipped with the norm defined by $\|\psi\|_\tau = \sup_{s\in[-\tau,0]}\|\psi(s)\|$, for $\psi \in \mathcal{PC}([-\tau,0],\mathbb{R}^n)$. For constant $\rho > 0$, define $\mathcal{B}(\rho) = \{x \in \mathbb{R}^n \mid \|x\| \le \rho\}$.

Consider the following nonlinear time-delay system subject to distributed-delay dependent impulses:

$$\begin{cases} \dot{x}(t) = f(t,x_t),\ t \in [t_{k-1},t_k), \\ \Delta x(t) = I_k(t, \int_{t-r_k}^{t} x(s)\mathrm{d}s),\quad t = t_k,\ k \in \mathbb{N}, \\ x_{t_0} = \psi, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $0 \le t_0 < t_1 < \cdots < t_k < \cdots$ with $\lim_{t\to\infty} t_k = \infty$, and $\Delta x(t) = x(t^+) - x(t^-)$. Here, we assume that x is right-continuous at each $t = t_k$, i.e., $x(t_k^+) = x(t_k)$. $x_t \in \mathcal{PC}([-\tau,0],\mathbb{R}^n)$ is defined as $x_t(s) = x(t+s)$ for $s \in [-\tau,0]$, where τ denotes the time-delay in the continuous dynamics of system (1). $r_k > 0$ represents the distributed delay in the impulse satisfying $r_k \le r \le \tau$ for all $k \in \mathbb{N}$. Assume $f : \mathbb{R}^+ \times \mathcal{PC}([-\tau,0],\mathcal{D}) \to \mathbb{R}^n$ and $I_k : \mathbb{R}^+ \times \mathcal{D} \to \mathbb{R}^n$, where $\mathcal{D} \subseteq \mathbb{R}^n$ is an open set, satisfy all the sufficient conditions in [21] so that system (1) admits a solution $x(t) := x(t,t_0,\psi)$ that exists on a maximal interval $[t_0-\tau, t_0+T)$ where $0 < T \le \infty$, and moreover, $f(t,0) = I_k(t,0) = 0$ for all $k \in \mathbb{N}$. Next, we further assume that, for some $\rho > 0$ and $\mathcal{B}(\rho) \subseteq \mathcal{D}$,

- (A_1) there exists a positive constant L_1 such that $\|f(t,\phi)\| \le L_1\|\phi\|_\tau$ for any $(t,\phi) \in \mathbb{R}^+ \times \mathcal{PC}([-\tau,0],\mathcal{B}(\rho))$;
- (A_2) there exists a positive constant L_2 such that $\|I_k(t,y) - I_k(t,z)\| \le L_2\|y-z\|$ for any $t \in \mathbb{R}^+$ and $y,z \in \mathcal{B}(\rho)$;
- (A_3) there exist positive constants $\underline{\sigma}$ and $\overline{\sigma}$ such that $\underline{\sigma} \le t_k - t_{k-1} \le \overline{\sigma}$ for all $k \in \mathbb{N}$, i.e., all the impulsive intervals are uniformly bounded;
- (A_4) there exists a nonnegative integer l such that $l\underline{\sigma} < r \le (l+1)\underline{\sigma}$, i.e., there are at most l impulses on each interval $[t_k - r_k, t_k)$.

Remark 1. Impulsive system (1) is equivalent to the following control system:

$$\dot{x} = f(t,x_t) + u(t), \tag{2}$$

with impulsive controller (IC)

$$u(t) = \sum_{k=1}^{\infty} I_k(t, \int_{t-r_k}^{t} x(s)\mathrm{d}s)\delta(t-t_k) \tag{3}$$

where $\delta(\cdot)$ is the Delta Dirac function. Recent results about delay-dependent impulsive control of time-delay systems were reported in [17], and the following form of delay-dependent impulses was considered:

$$x(t_k) = \Gamma_k x(t_k - \varsigma_k), \tag{4}$$

where $\Gamma_k \in \mathbb{R}^n$ and ς_k denotes the discrete delay in the impulse. Rewrite (4) as $\Delta x(t_k) = -x(t_k^-) + \Gamma_k x(t_k - \varsigma_k)$, then the corresponding IC is

$$u(t) = \sum_{k=1}^{\infty}[-x(t) + \Gamma_k x(t-\varsigma_k)]\delta(t-t_k), \tag{5}$$

which depends not only on the states at a history instant (i.e., $x(t_k - \varsigma_k)$) but also on the states at the impulsive time (i.e., $x(t_k^-)$). Therefore, sufficient conditions obtained in [17] could guarantee the IC (5) to stabilize the time-delay system, but the authors cannot make conclusion that the delayed states contribute to the systems stability. However, it can be seen that IC (3) relies purely on the distributed-delay dependent states, i.e., the distributed delays in IC (5) play a key role in stabilization of the nonlinear system.

The objective of this technical note is to use Lyapunov-Razumikhin method to establish exponential stability criteria for impulsive system (1). Next, we shall list exponential stability and Lyapunov function related definitions, respectively.

Definition 1. The trivial solution of system (1) is said to be exponentially stable (ES), if there exist positive constants ρ_0, M and α such that

$$\|x(t)\| \le M\|\psi\|_\tau e^{-\alpha(t-t_0)},\quad t \ge t_0, \tag{6}$$

for any $\psi \in \mathcal{PC}([-\tau,0],\mathcal{B}(\rho_0))$. Furthermore, if (6) holds for any $\psi \in \mathcal{PC}([-\tau,0],\mathbb{R}^n)$, then the trivial solution of (1) is said to be globally exponentially stable (GES).

Definition 2. Function $V : [t_0-\tau,\infty) \times \mathcal{B}(\rho) \to \mathbb{R}^+$ is said to belong to the class of ν_0 if

- V is continuous on $[t_{k-1},t_k) \times \mathcal{B}(\rho)$, and for each $x \in \mathbb{R}^n$ and $t \in [t_{k-1},t_k)$, $\lim_{(t,y)\to(t_k^-,x)} V(t,y) = V(t_k^-,x)$ exists;
- $V(t,x)$ is locally Lipschitz in $x \in \mathcal{B}(\rho)$, and $V(t,0) = 0$ for all $t \ge t_0$.

Definition 3. Given a function $V \in \nu_0$, the upper right-hand derivative $D^+V(t,\phi(0))$ along the solution of system (1) is defined by $D^+V(t,\phi(0)) = \limsup_{h\to 0}\frac{1}{h}[V(t+h,\phi(0)+hf(t,\phi)) - V(t,\phi(0))]$, where $(t,\phi) \in [t_0,\infty) \times \mathcal{PC}([-\tau,0],\mathcal{B}(\rho))$.

3. Stabilization results

In this section, we first construct an exponential stability criterion for system (1).

Theorem 1. *Suppose assumptions (A_1)–(A_4) are satisfied, and there exist a function $V \in \nu_0$, and positive constants $c_1, c_2, p, c, q, K_1, K_2$ and ν such that*

- (i) $c_1\|x\|^p \le V(t,x) \le c_2\|x\|^p$ *for all* $(t,x) \in [t_0-\tau,\infty) \times \mathcal{B}(\rho)$;
- (ii) $D^+V(t,\phi(0)) \le cV(t,\phi(0))$ *for all* $t \in \{t \in [t_0,\infty) \mid t \ne t_k, k \in \mathbb{N}\}$ *and* $\phi \in \mathcal{PC}([-\tau,0],\mathcal{B}(\rho))$, *whenever* $V(t+s,\phi(s)) \le qV(t,\phi(0))$ *for all* $s \in [-\tau,0]$;
- (iii) $V(t,x+y) \le K_1V(t,x) + K_2V(t,y)$ *for all* $t = t_k$ *and* $x,y \in \mathcal{B}(\rho)$ *satisfying* $x+y \in \mathcal{B}(\rho)$;
- (iv) $V(t,x+I_k(t,r_kx)) \le \nu V(t^-,x)$ *for all* $t = t_k$ *and* $x \in \mathcal{B}(\frac{\rho}{1+rL_2})$;
- (v) $q > \{K_1\nu + K_2\frac{c_2}{c_1}[r^2L_2(L_1+lL_2)]^p\}^{-1} > e^{c\overline{\sigma}}$,

then the trivial solution of system (1) is ES.

Proof. Let $d_1 = K_1\nu$ and $d_2 = K_2\frac{c_2}{c_1}[r^2L_2(L_1+lL_2)]^p$. From condition (v), we can find a small enough constant α such that

$$q > \frac{e^{\alpha\overline{\tau}}}{d_1 + d_2e^{\alpha\overline{\tau}}} > \frac{1}{d_1+d_2e^{\alpha\overline{\tau}}} > e^{(c+\alpha)\overline{\sigma}}, \tag{7}$$