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A macro traffic flow model with probability distribution function

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A B S T R A C T

In this paper, we introduce three probability distribution functions into the dynamic equation and propose a macro traffic flow model to investigate the impacts of the probability distribution functions on the evolutions of traffic flow under three typical states (i.e., uniform flow, shock, rarefaction wave, and small perturbation). The numerical results indicate that the probability distribution functions do not change the density and speed distributions of uniform flow, produce a two-layer shock but have no prominent effects on rarefaction wave, and have little effect on small perturbation.

1. Introduction

To date, various traffic problems (e.g., jam, accident, pollution, etc.) have become very serious, so researchers have proposed many models to study and explain the formation mechanisms of the traffic problems, and some management strategies to solve or relieve the traffic problems [1,2]. The traffic flow models can simply be sorted into macro ones [3–28] and micro ones [29–63]. The macro models use macro variables (e.g., density, speed, flow) to study the distribution, evolution and propagation of traffic flow and interpret the mechanisms of the complex traffic phenomena (e.g., jam), where the mathematical models consist of some partial differential equations (PDEs) defined by speed and density [3–28]. The micro models use some micro variables (e.g., each individual's acceleration, speed and headway) to explore the driving behavior and the corresponding complex traffic phenomena and explain the mechanisms of the traffic phenomena, where the mathematical models consist of a series of ordinary differential equations (ODEs) defined by each individual's acceleration [29–57].

The models [3–57] can describe many traffic phenomena and explain their mechanisms, but they cannot be used to directly study the impacts of various stochastic factors since they did not explicitly consider the factors. In fact, many stochastic factors widely exist in a traffic system (especially the urban traffic system). For example, each driver may randomly adjust his/her driving behavior (e.g., speed) due to some specific conditions (e.g., distraction), which shows that the stochastic factors should be considered in a traffic flow model (especially in a driving behavior model). To study this topic, Tang et al. [58–60] introduced the stochastic factor into a traffic flow model and proposed three models with stochastic factor to study the influences of the stochastic factor on the driving behavior and the density distribution. The numerical results indicate that the impacts are dependent on the driver's individual attribution. However, Tang et al. [58–60] did not define the distribution function of each stochastic factor, so the models do not perfectly reproduce the impacts of the distribution function on traffic flow. To conquer this shortcoming, Ou et al. [61] defined the distribution functions of three stochastic variables, and then proposed a car-following model to explore the effects of the distribution functions on uniform flow, shock, rarefaction wave, and small perturbation, but they did not explore the impacts from the macroscopic perspective. In this paper, we transform the micro variables [61] into the macro ones, and develop a macro traffic flow model to explore the effects of the probability distribution functions on traffic flow. This paper is organized as follows: the traffic flow model with the probability distribution functions is developed in Section 2; some numerical tests are carried out to explore the influences of the probability distribution functions on traffic flow in Section 3; and some conclusions are summarized in Section 4.

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2. Model

The first macro traffic flow was independently proposed by Lighthill and Whitham [2] and Richards [3] (called as the LWR model), where the control equation can be formulated as follows:

$$\rho_t + \left(\rho v_e(\rho)\right)_x = 0, \tag{1}$$

where ρ is the density; $v_e(\rho)$ is the equilibrium speed that satisfied the following conditions:

(1) $v_e(\rho)$ is a decreasing function of ρ and $v_e(\rho_j) = 0$, where ρ_j is the jam density.
(2) As the function of ρ, the flux $\rho v_e(\rho)$ is a concave function.

Eq. (1) can reproduce the formation and propagation of shock, but it cannot be used to explore the non-equilibrium traffic flow since the speed cannot deviate from the equilibrium speed. To overcome this shortcoming, researchers substituted an acceleration for the equilibrium speed in Eq. (1) and proposed many high-order models [3–28]. The high-order models can be divided into the density-gradient (DG) models and speed-gradient (SG) models, where the first DG model can be formulated as follows [3]:

$$\begin{cases} \rho_t + (\rho v_e(\rho))_x = 0 \\ v_t + v v_x = \dfrac{v_e(\rho) - v}{T} - \dfrac{c^2(\rho)}{\rho}\rho_x \end{cases}, \tag{2}$$

where $c(\rho) > 0$ is the sonic speed.

The simplest SG model can be formulated as follows [6]:

$$\begin{cases} \rho_t + (\rho v_e(\rho))_x = 0 \\ v_t + v v_x = \dfrac{v_e(\rho) - v}{T} + c_0 v_x \end{cases}, \tag{3}$$

where $c_0 > 0$ is the propagation speed of small perturbation.

Eqs. (1)–(3) do not consider any stochastic factors, so they cannot be used to study various complex traffic phenomena caused by the stochastic factors. To study the impacts of some stochastic factors (e.g., the driver's perceived errors) on traffic flow, Ou et al. [61] developed a car-following model with the probability distribution functions, i.e.,

$$a_n = \kappa\left(V(\Delta x_n + \xi_n) - v_n\right) + \lambda(\Delta v_n + \eta_n) + \varepsilon_n, \tag{4}$$

where $a_n, \Delta x_n, \Delta v_n$ are the nth driver's acceleration, headway and relative speed, respectively; κ, λ are two reaction coefficients; V is the optimal speed; $\xi_n, \eta_n, \varepsilon_n$ are three stochastic variables.

Applying the same method [6], we can propose a macro traffic flow model with consideration of the probability distribution functions, where the control equation can be formulated as follows:

$$\begin{cases} \rho_t + (\rho v_e(\rho))_x = 0 \\ v_t + v v_x = \dfrac{v_e(\rho + \xi) - v}{T} + c_0(v_x + \eta) + \varepsilon \end{cases}, \tag{5}$$

where ξ is the perceived error of ρ; η is the perceived error of v_x; ε is the perceived error of $\hat{a}$ ($\hat{a}$ is the acceleration calculated by Eq. (3)).

Using the same method [61], we can here assume that the parameters ξ, η, ε follow the normal distribution and that their probability densities and standard variances can be formulated as follows[1]:

$$f_\xi(\xi) = \frac{1}{\sqrt{2\pi}\sigma_\xi}\exp\left(-\frac{\xi^2}{2\sigma_\xi^2}\right), \tag{6}$$

$$f_\eta(\eta) = \frac{1}{\sqrt{2\pi}\sigma_\eta}\exp\left(-\frac{\eta^2}{2\sigma_\eta^2}\right), \tag{7}$$

$$f_\varepsilon(\varepsilon) = \frac{1}{\sqrt{2\pi}\sigma_\varepsilon}\exp\left(-\frac{\varepsilon^2}{2\sigma_\varepsilon^2}\right), \tag{8}$$

$$g_\xi(\rho) = \frac{\beta_1\alpha|\rho| + \beta_2}{u_{0.975}}, \tag{9}$$

$$g_\eta(v_x) = \frac{\beta_1\alpha|v_x| + \beta_2}{u_{0.975}}, \tag{10}$$

$$g_\varepsilon(\hat{a}) = \frac{\beta_1\alpha|\hat{a}| + \beta_2}{u_{0.975}}, \tag{11}$$

where $f_\xi, f_\eta, f_\varepsilon$ are three probability density functions; $g_\xi, g_\eta, g_\varepsilon$ are the corresponding standard variances; $\beta_1 \geq 1, \beta_2 \geq 0, 0 < \alpha < 1$ are three parameters; $u_{0.975}$ is 1.96.

Based on the above discussions, the macro traffic flow model with the probability distribution can be expressed by Eqs. (11)–(16). Comparing with the existing macro models, the proposed model can be used to study the influences of the probability distribution on traffic flow, so it is better than the existing macro models.

3. Simulation

In this section, we explore what phenomena the proposed model can reproduce. It is difficult to obtain the analytical solution of Eq. (11), so we use numerical tests to study the influences of the probability distribution functions on traffic flow. Eq. (11) has different numerical schemes, but the schemes have no qualitative impacts on the numerical results, so we here utilize the upwind difference to discretize Eq. (11), i.e.,

$$\rho_i^{j+1} = \rho_i^j + \frac{\Delta t}{\Delta x}\rho_i^j\left(v_i^j - v_{i+1}^j\right) + \frac{\Delta t}{\Delta x}v_i^j\left(\rho_{i-1}^j - \rho_i^j\right), \tag{12}$$

if $v_i^j \leq c_0$

$$v_i^{j+1} = v_i^j + \frac{\Delta t}{\Delta x}\left(c_0 - v_i^j\right)\left(v_{i+1}^j - v_i^j\right) + \Delta t\left(c_0\eta_i^j + \varepsilon_i^j\right) + \frac{\Delta t}{T}\left(v_e\left(\rho_i^j + \xi_i^j\right) - v_i^j\right), \tag{13a}$$

and otherwise

$$v_i^{j+1} = v_i^j + \frac{\Delta t}{\Delta x}\left(c_0 - v_i^j\right)\left(v_i^j - v_{i-1}^j\right) + \Delta t\left(c_0\eta_i^j + \varepsilon_i^j\right) + \frac{\Delta t}{T}\left(v_e\left(\rho_i^j + \xi_i^j\right) - v_i^j\right), \tag{13b}$$

where i, j are respectively the spatial and time indexes; $\Delta x = 100$ m is the space step; $\Delta t = 1$ s is the time step.

Like Ref. [6], we define the equilibrium speed as follows [62]:

$$v_e(\rho) = v_f\left(1\Big/\left(1 + \exp\left(\frac{\rho/\rho_j - 0.25}{0.06}\right)\right) - 3.72\times10^{-6}\right), \tag{14}$$

where v_f is the free flow speed.

Other related parameters are defined as follows:

$$\begin{aligned} &\alpha = 0.1, \quad \beta_1 = 1.0, \quad \beta_2 = 0, \quad T = 10\text{ s}, \\ &v_f = 30\text{ m/s}, \quad c_0 = 10\text{ m/s}, \quad \rho_j = 0.2\text{ veh/m}. \end{aligned} \tag{15}$$

In the following numerical tests, the initial speed is set as follows:

[1] Note: the related parameters in Eqs. (6)–(11) are different from the ones in Ref. [61], but we do not consider the differences in this paper.