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Epsilon-strongly graded rings, separability and semisimplicity

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We introduce the class of epsilon-strongly graded rings and show that it properly contains both the class of strongly graded rings and the class of unital partial crossed products. We determine precisely when an epsilon-strongly graded ring is separable over its principal component. Thereby, we simultaneously generalize a result for strongly group graded rings by Năstăsescu, Van den Bergh and Van Oystaeyen, and a result for unital partial crossed products by Bagio, Lazzarin and Paques. We also show that the class of unital partial crossed products appears in the class of epsilon-strongly graded rings in a fashion similar to how the classical crossed products present themselves in the class of strongly graded rings. Thereby, we obtain, in the special case of unital partial crossed products, a short proof of a general result by Dokuchaev, Exel and Simón concerning when graded rings can be presented as partial crossed products. We also provide some interesting classes of examples of separable epsilon-strongly graded rings, with finite as well as infinite grading groups. In

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particular, we obtain an answer to a question raised by Le Bruyn, Van den Bergh and Van Oystaeyen in 1988.

1. Introduction

Let S be an associative ring equipped with a non-zero multiplicative identity element 1. Let S/R be a ring extension. By this we mean that R is a subring of S containing 1. Recall that S/R is called *separable* if the multiplication map $m : S \otimes_R S \to S$ is a splitting epimorphism of R-bimodules. Equivalently, this can be formulated by saying that there is $x \in S \otimes_R S$ satisfying $m(x) = 1$ and that, for every $s \in S$, the relation $sx = xs$ holds. In that case, x is called a *separability element* of $S \otimes_R S$. Separable ring extensions are a natural generalization of the classical separability condition for algebras over fields which in turn is a generalization of separable field extensions (see e.g. [7]). Năstăsescu, Van den Bergh and Van Oystaeyen [20] have generalized this even further by introducing the notion of a *separable functor*. They show that a ring extension is separable precisely when the associated restriction functor is separable. A lot of work has been devoted to the question of when ring extensions are separable (see e.g. [1], [14], [2], [3], [7], [9], [10], [15], [16], [19], [20], [23] and [24]). One reason for this intense interest is that some properties of the ground ring R, such as semisimplicity and hereditarity, are automatically inherited by S (see e.g. [20]).

In the context of group graded rings, necessary and sufficient conditions for separability has been obtained in two different cases (see Theorem 1 and Theorem 2 below). Indeed, let G be a group with identity element e. Let S be *graded* by G. Recall that this means that, for all $g, h \in G$, there is an additive subgroup S_g of S such that $S = \oplus_{g \in G} S_g$ and $S_g S_h \subseteq S_{gh}$. The subring $R = S_e$ is called the *principal component* of S.

In the first case, S is *strongly graded*. Recall that this means that $S_g S_h = S_{gh}$, for all $g, h \in G$. This makes each S_g, for $g \in G$, an invertible R-bimodule which implies that there is a unique ring automorphism $\beta_g : Z(R) \to Z(R)$ such that $\beta_g(r)s = sr$, for $r \in Z(R)$ and $s \in S_g$ (see [19] or e.g. Definition 10 and Proposition 13). If G is finite, then the trace function $\mathrm{tr}_\beta : Z(R) \to Z(R)$ is defined by $\mathrm{tr}_\beta(r) = \sum_{g \in G} \beta_g(r)$, for $r \in Z(R)$.

Theorem 1 *(Năstăsescu, Van den Bergh and Van Oystaeyen [20]). If S is strongly graded by G, then S/R is separable if and only if G is finite and $1 \in \mathrm{tr}_\beta(Z(R))$.*

In the second case, S is a *unital partial crossed product* of G over R. Recall that a *unital twisted partial action* of G on R is a triple

$$\alpha = (\{D_g\}_{g \in G}, \{\alpha_g\}_{g \in G}, \{w_{g,h}\}_{(g,h) \in G \times G})$$