ARTICLE IN PRESS
YJDEQ:9354

Available online at www.sciencedirect.com
ScienceDirect

ELSEVIER

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Convergence in time-periodic quasilinear parabolic equations in one space dimension ☆

Bendong Lou

Mathematics and Science College, Shanghai Normal University, Shanghai 200234, China

Received 13 June 2017; revised 31 January 2018

Abstract

We consider time-periodic quasilinear parabolic equations in the domain $\{(t,x) \in \mathbb{R}^2 \mid 0 < x < r(t), t > 0\}$, where the right boundary $r(t)$ of the spatial interval is a positive function which might be periodic, or asymptotically periodic, or a function tending to infinity, or infinity. We show that, in the first case (that is, $r(t)$ is a periodic function), any bounded solution of the equations converges as $t \to \infty$ to a periodic one; in the other three cases, any positive bounded solution converges as $t \to \infty$ to a nonnegative periodic one. Using such a result, we study the long time dynamics of the initial-boundary value problem on the half line, as well as the Stefan free boundary problem, of a general heterogeneous reaction–diffusion equation. Also, we use the convergence result to study the long time dynamics of the initial-boundary value problem for a time-periodic (mean) curvature flow equation.

MSC: 35K55; 35K20; 35B40; 35R35

Keywords: Nonlinear parabolic equation; Time-periodic solution; Initial-boundary value problem; Asymptotic behavior; Free boundary problem

1. Introduction

In this paper we first consider the convergence of bounded solutions of the following quasi-linear parabolic equation in one space dimension

☆ This research was partly supported by NSFC (No. 11671262).
E-mail address: lou@shnu.edu.cn.

https://doi.org/10.1016/j.jde.2018.05.025

$$u_t = d(t, x, u, u_x)u_{xx} + f(t, x, u, u_x), \quad x \in (0, r(t)),\ t > 0, \tag{1.1}$$

where $d, f \in C^1$ with $d > 0$ are periodic in t with a common period T, the right boundary $r(t)$ of the spatial interval is a positive function (see details below). We adopt the homogeneous Dirichlet boundary conditions

$$u(t, 0) = 0, \quad u(t, r(t)) = 0, \quad t > 0. \tag{1.2}$$

When $r(t) = \infty$, the second boundary condition is neglected automatically. So our problem to be studied is (1.1)–(1.2).

We will present some convergence results for bounded solutions. Such kind of results are of special importance in the study of qualitative properties of the solutions. Let us recall some known results in this field for equations in one space dimension.

1. *Autonomous equations in a fixed bounded interval*. When d, f are independent of t, the equation is an autonomous one. If, in addition, $r(t)$ is a positive constant, then the problem is a rather simpler one which has been well studied. For example, Zelenyak [34] (see also Matano [23,25]) proved that any bounded solution of such a simple problem converges as $t \to \infty$ to a stationary one.

2. *Autonomous reaction–diffusion equations in unbounded intervals*. Consider

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R},\ t > 0, \tag{1.3}$$

with f locally Lipschitz and $f(0) = 0$. Du and Matano [16] proved that any bounded nonnegative solution starting from a compactly supported initial data converges to a stationary one. Chen et al. [8] proved a similar result for the problem on the half line $[0, \infty)$ (though they considered only the equations with bistable type of nonlinearity, their argument applies to general equations like (1.3)). When the initial data is not a compactly supported one, say, it is in $C_0(\mathbb{R})$ (the space of continuous functions with limits at $\pm\infty$ equaling to 0), Matano and Polacik [27] obtained similar convergence result very recently.

3. *Time periodic equations in fixed bounded interval*. Chen and Matano [10] considered the equation $u_t = u_{xx} + f(t, u)$ with f being periodic in t in a fixed interval $(0, \bar{r})$, and proved that any bounded solution of this equation with Dirichlet, Neumann or periodic boundary conditions converges to a periodic one. Later Brunovsky et al. [4] extended such a result to the general equation (1.1).

4. *Reaction–diffusion equations in variable intervals*. Variable intervals appear in some applied fields, to say, in the free boundary problems. Du and Lin [14] and Du and Lou [15] studied the equation (1.3) in the interval $(0, r(t))$ with r satisfying the Stefan free boundary conditions:

$$u(t, r(t)) = 0, \quad r'(t) = -u_x(t, r(t)), \quad t > 0. \tag{1.4}$$

They used such a problem to model the spreading of a new or invasive species, with u denoting the population density and the free boundary $r(t)$ representing the (monotone increasing) expanding front of the species. Du and Lou [15] proved that, for general $f(u)$ satisfying $f(0) = 0$, any bounded positive solution of this problem converges to a stationary one. In the last few years, using the free boundary condition as in (1.4), some authors studied various reaction–diffusion equations (in advective or spatially heterogeneous environments) (cf. [13,18,19,31]), or time pe-

Please cite this article in press as: B. Lou, Convergence in time-periodic quasilinear parabolic equations in one space dimension, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.05.025