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Journal of Differential Equations

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Critical points of solutions to a quasilinear elliptic equation with nonhomogeneous Dirichlet boundary conditions ☆

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Received 25 December 2017; revised 10 May 2018

Abstract

In this paper, we mainly investigate the critical points associated to solutions u of a quasilinear elliptic equation with nonhomogeneous Dirichlet boundary conditions in a connected domain Ω in $\mathbb{R}^2$. Based on the fine analysis about the distribution of connected components of a super-level set $\{x \in \Omega : u(x) > t\}$ for any $\min_{\partial\Omega} u(x) < t < \max_{\partial\Omega} u(x)$, we obtain the geometric structure of interior critical points of u. Precisely, when Ω is simply connected, we develop a new method to prove $\Sigma_{i=1}^{k} m_i + 1 = N$, where $m_1, \cdots, m_k$ are the respective multiplicities of interior critical points $x_1, \cdots, x_k$ of u and N is the number of global maximal points of u on $\partial\Omega$. When Ω is an annular domain with the interior boundary γ_I and the external boundary γ_E, where $u|_{\gamma_I} = H,\ u|_{\gamma_E} = \psi(x)$ and $\psi(x)$ has N local (global) maximal points on γ_E. For the case $\psi(x) \geq H$ or $\psi(x) \leq H$ or $\min_{\gamma_E} \psi(x) < H < \max_{\gamma_E} \psi(x)$, we show that $\Sigma_{i=1}^{k} m_i \leq N$ (either $\Sigma_{i=1}^{k} m_i = N$ or $\Sigma_{i=1}^{k} m_i + 1 = N$).

MSC: 35J93; 35J25; 35B38

Keywords: A quasilinear elliptic equation; Critical point; Multiplicity; Multiply connected domain

☆ The work is supported by National Natural Science Foundation of China (No. 11401307, No. 11401310), High level talent research fund of Nanjing Forestry University (G2014022) and Postgraduate Research & Practice Innovation Program of Jiangsu Province (KYCX17_0321). The second author is sponsored by Qing Lan Project of Jiangsu Province.
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https://doi.org/10.1016/j.jde.2018.05.031
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1. Introduction and main results

In this paper we mainly investigate the interior critical points of solutions to the following a quasilinear elliptic equation

$$Lu = \sum_{i,j=1}^{2} a_{ij}(\nabla u)\frac{\partial^2 u}{\partial x_i \partial x_j} = 0 \quad \text{in } \Omega, \tag{1.1}$$

where Ω is a bounded, smooth and connected domain in $\mathbb{R}^2$, a_{ij} is smooth and L is uniformly elliptic in Ω.

The subject of critical points is a significant research topic for solutions of elliptic equations. Until now, there are many results about the critical points. In 1992 Alessandrini and Magnanini [1] studied the geometric structure of the critical set of solutions to a semilinear elliptic equation in a planar nonconvex domain, whose boundary is composed of finite simple closed curves. They deduced that the critical set is made up of finitely many isolated critical points. In 1994, Sakaguchi [19] considered the critical points of solutions to an obstacle problem in a planar, bounded, smooth and simply connected domain. He showed that if the number of critical points of the obstacle is finite and the obstacle has only N local (global) maximum points, then the inequality $\Sigma_{i=1}^{k} m_i + 1 \leq N$ (the equality $\Sigma_{i=1}^{k} m_i + 1 = N$) holds for the critical points of one solution in the noncoincidence set, where $m_1, m_2, \cdots, m_k$ are the multiplicities of critical points $x_1, x_2, \cdots, x_k$ respectively. In 2012 Arango and Gómez [3] considered critical points of the solutions to a quasilinear elliptic equation with Dirichlet boundary condition in strictly convex and nonconvex planar domains respectively. If the domain is strictly convex and u is a negative solution, they proved that such a critical point set has exactly one nondegenerate critical point. Moreover, they obtained the similar results of a semilinear elliptic equation in a planar annular domain, whose boundary has nonzero curvature. See [2,5,6,9–12,14–17] for related results.

Concerning the Neumann and Robin boundary value problems, there exist a few results about the critical points of solutions to elliptic equations. In 1990, Sakaguchi [18] proved that a solution of Poisson equation with Neumann or Robin boundary condition has exactly one critical point in a planar domain. In 2017, Deng, Liu and Tian [8] showed the nondegeneracy and uniqueness of the critical point of a solution to prescribed constant mean curvature equation with Neumann or Robin boundary condition in a smooth, bounded and strictly convex domain Ω of $\mathbb{R}^n (n \geq 2)$.

For the higher dimensional cases. Under the assumption of the existence of a semi-stable solution of Poisson equation $-\Delta u = f(u)$, Cabré and Chanillo [4] showed that the solution u has exactly one nondegenerate critical point in bounded, smooth and convex domains of $\mathbb{R}^n (n \geq 2)$. Deng, Liu and Tian [7] investigated the geometric structure of critical points of solutions to mean curvature equations with Dirichlet boundary condition and showed that the critical point set K has exactly one nondegenerate critical point in a strictly convex domain of $\mathbb{R}^n (n \geq 2)$ and K has (respectively, has no) a rotationally symmetric critical closed surface S in a concentric (respectively, an eccentric) spherical annulus domain of $\mathbb{R}^n (n \geq 3)$.

However, as we know, there is few work on the critical points of solutions to quasilinear elliptic equations with nonhomogeneous Dirichlet boundary conditions. The goal of this paper is to study the critical points of solutions to a quasilinear elliptic equation with nonhomogeneous Dirichlet boundary conditions. Our main results are as follows.

Please cite this article in press as: H. Deng et al., Critical points of solutions to a quasilinear elliptic equation with nonhomogeneous Dirichlet boundary conditions, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.05.031