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Global strong solutions to abstract quasi-variational evolution equations

Masahiro Kubo [a,*,1], Noriaki Yamazaki [b,2]

[a] Faculty of Systems Engineering, Wakayama University, Wakayama, 640-8510, Japan

[b] Department of Mathematics, Faculty of Engineering, Kanagawa University, Yokohama, 221-8686, Japan

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Abstract

We prove the existence of a time-global strong solution of a class of abstract quasi-variational evolution equations and apply the abstract result to concrete parabolic variational and quasi-variational inequalities.

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Keywords: Variational inequalities; Quasi-variational inequalities; Subdifferential operators; Evolution equations

1. Introduction

We study an evolution equation in a real Hilbert space H of the following form:

$$u'(t) + \partial\varphi(t, u; u(t)) \ni 0, \quad 0 < t < T. \tag{1.1}$$

Here, for a subset $\mathscr{K}$ of $L^2(0, T; H)$ (see (2.1)), a function

$$\varphi : [0, T] \times \mathscr{K} \times H \to [0, +\infty]$$

* Corresponding author.
E-mail address: masahiro@sys.wakayama-u.ac.jp (M. Kubo).

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is given in a way in which $\varphi(t, v; \cdot) : H \to [0, +\infty]$ is a proper, l.s.c. (lower-semicontinuous) and convex function for each $(t, v) \in [0, T] \times \mathscr{K}$. The symbol $\partial\varphi(t, v; \cdot)$ stands for the subdifferential of $\varphi(t, v; \cdot)$ for each fixed (t, v).

In this paper, we prove the existence of a time-global solution of the equation (1.1) satisfying an initial condition:

$$u(0) = u_0, \tag{1.2}$$

assuming appropriate conditions for the dependence of $\varphi(t, v; \cdot)$ on t and v. No monotonicity property of the map $v \mapsto \varphi(t, v; \cdot)$ is posed.

Equation (1.1) models quasi-variational inequalities of the following form:

$$\begin{cases} u(t) \in K(t, u), \\ (u' - f, u - z) + \int_\Omega \boldsymbol{a}(u, \nabla u) \cdot \nabla(u - z) dx \le 0 \ \text{ for all } z \in K(t, u). \end{cases} \tag{1.3}$$

Here, we are given a family $K(t, v)$ of convex sets depending on $t \in [0, T]$ and $v \in L^2(0, T; H)$. This type of problems arise in mathematical models of various areas, e.g., phase transitions, filtration problems and so on. The literature of the theory as well as applications is so vast that we will not exhaust it but refer, in the following description, to a part closely related to the present study.

The result of the present paper synthesizes those of our previous studies [12,16–19,24] and those of Kano et al. [4–8] (see Remark 2.1). In the former papers, variational inequalities arising in saturated–unsaturated filtration problems with explicit time-dependent constraint $K(t)$ instead of $K(t, v)$ in (1.3) were studied. While in the latter, the implicit constraint $u(t) \in K(t, u)$ was considered for a limited type differential operator such as $\nabla \cdot \boldsymbol{a}(\nabla u)$. We note that in filtration problems, for instance, the dependence of $\nabla \cdot \boldsymbol{a}(u, \nabla u)$ on u models the effect of gravitational force and hence plays an important role. We notice that one of the delicate points for this synthesis is to employ the set $\mathscr{K}$ defined by (2.1) (see Remarks 2.2, 2.8, 4.1).

We consider strong solutions, that is, solutions which have strong time derivatives as H valued functions. We remark that the existence of a weak solution was studied by Stefanelli [23], Kenmochi and Stefanelli [14], and Kenmochi and Niezgódka [13]. For earlier works on parabolic quasi-variational inequalities, we refer to Mignot and Puel [20] and references therein. We also refer to Rodrigues and Santos [22] where a concrete problem for a superconductivity model was studied by a penalty method.

We state in the next section the main result (Theorem 2.4) asserting the existence of a global strong solution to {(1.1), (1.2)} and prove it by using a property for an auxiliary problem proved in the subsequent Section 3.

The method in Section 3 is originated in the theory of time-dependent subdifferential evolution equations developed by Kenmochi [9,10] and Yamada [25]. The key role is played by the chain rule and energy inequality as described in Proposition 3.3. In fact, this characterizes the theory itself as shown in [15]. Our main interest in application is variational and quasi-variational inequalities, whereas the subdifferential operator theory can be applied to various important nonlinear PDEs without constraint, as studied by Ôtani [21] and Akagi [1].

Finally, in Section 4, we apply the abstract Theorem 2.4 to concrete problems such as (1.3), which will exemplify the synthesis of our previous studies referred above.

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