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Complex symmetric differential operators on Fock space

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Abstract

The space of entire functions which are integrable with respect to the Gaussian weight, known also as the Fock space, is one of the preferred functional Hilbert spaces for modeling and experimenting harmonic analysis, quantum mechanics or spectral analysis phenomena. This space of entire functions carries a three parameter family of canonical isometric involutions. We characterize the linear differential operators acting on Fock space which are complex symmetric with respect to these conjugations. In parallel, as a basis of comparison, we discuss the structure of self-adjoint linear differential operators. The computation of the point spectrum of some of these operators is carried out in detail.

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1. Introduction

1.1. Complex symmetric operators

The foundational works of Glazman and Zihar [9,10,22] mark the beginning of the theory of unbounded complex symmetric operators. The non-symmetric or non-self-adjoint differential operators treated by them did not enter into the classical framework initiated by von Neumann, nor in the dissipative operator class studied around mid XX-th century by Keldys, Krein and Livsic. What makes complex symmetric differential operators special is the fact that they carry a weak form of spectral decomposition theorem (discovered much earlier in the case of matrices by Takagi). While the main path of research was continued after Glazman on a Hamiltonian mechanics path, in the context of mature by now operator theory in spaces with an indefinite metric [2,18], the more recent theory of non-hermitian quantum mechanics has naturally enlarged the class of examples of complex symmetric differential operators [3,4,14,23]. Pertinent studies of complex symmetric boundary conditions for linear differential operators on an interval are also well known [17,19]. Adding to these examples classes of structured matrices (such as Toeplitz or Hankel) or integral operators which carry a canonical complex symmetry, we are contemplating today a vast territory, escaping the orthodox path of spectral analysis, but by no means less captivating and relevant for applications. A systematic study, at the abstract level, of complex symmetric operators was undertaken in [7,8]. Applications to mathematical physics and connections to the theory of operators in an indefinite metric space can be found in the survey [6].

The present article narrows down to the precise question of classifying linear differential operators acting on Fock space, which are complex symmetric with respect to a family of natural conjugations. These conjugations include as a special case the famous $\mathcal{PT}$-symmetry. Besides tedious but elementary identities at the level of formal series, the main complication of our study resides in the identification of the proper domains of the unbounded linear operators in question.

We start by recalling some basic definitions. Throughout this article $\mathcal{H}$ is a separable complex Hilbert space. The domain of an unbounded linear operator T is $\mathrm{dom}(T)$. For two unbounded operators A, B, the notation $A \preceq B$ means that B is an *extension* of A or that A is a *restriction* of B on $\mathrm{dom}(A)$, namely $\mathrm{dom}(A) \subseteq \mathrm{dom}(B)$ and $Ax = Bx$ for every $x \in \mathrm{dom}(A)$. If $A \preceq B$ and $B \preceq A$, then we write $A = B$. Furthermore, if C, D are two bounded operators on $\mathcal{H}$, then we define the operator CAD by $(CAD)f = C(A(Df))$ with the domain $\mathrm{dom}(CAD)$ including all elements $f \in \mathcal{H}$ for which $Df \in \mathrm{dom}(A)$.

Definition 1.1. An anti-linear operator $\mathcal{C} : \mathcal{H} \to \mathcal{H}$ is called a *conjugation* if it is both involutive and isometric.

In the presence of a conjugation $\mathcal{C}$, the inner product of the underlying Hilbert space induces a bounded, complex bilinear symmetric form

$$[x, y] = \langle x, \mathcal{C}y\rangle, \quad x, y \in \mathcal{H}.$$

Symmetry of linear transforms with respect to this bilinear form is the main subject of our study.

Definition 1.2. Let $T : \mathrm{dom}(T) \subseteq \mathcal{H} \to \mathcal{H}$ be a closed, densely defined, linear operator and $\mathcal{C}$ a conjugation. We say that T is

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