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# Vanishing vertical viscosity limit of anisotropic Navier–Stokes equation with no-slip boundary condition

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**Abstract**

In this paper, we consider the zero viscosity limit of the anisotropic incompressible Navier–Stokes equations with no-slip boundary condition (see (1.1)) in $R^2_+$. We prove that there exist $T$ independently on $\varepsilon$ such that the strong solutions of (1.1) convergence to the solution of (1.2) away from the boundary and to solution of (1.3) near the boundary in $L^\infty((0,T), L^2\cap L^\infty(R^2_+))$ when the vertical viscosity vanish, provided that the initial velocity is regular enough and we obtain the optimal convergence rate.


## 1. Introduction

In this paper, we consider the vanishing vertical viscosity limit for the following anisotropic incompressible Navier–Stokes equations with the no-slip boundary condition in the half space $R^2_+$:

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + v^\varepsilon \partial_y u^\varepsilon + \partial_x p^\varepsilon - \partial_{xx} u^\varepsilon - \varepsilon^2 \partial_{yy} u^\varepsilon = 0, \\ \partial_t v^\varepsilon + u^\varepsilon \partial_x v^\varepsilon + v^\varepsilon \partial_y v^\varepsilon + \partial_y p^\varepsilon - \partial_{xx} v^\varepsilon - \varepsilon^2 \partial_{yy} v^\varepsilon = 0, \\ \partial_x u^\varepsilon + \partial_y v^\varepsilon = 0, \\ v^\varepsilon(t,x,0) = 0, u^\varepsilon(t,x,0) = 0, \\ (u^\varepsilon, v^\varepsilon)(0,x,y) = (u_0, v_0). \end{cases} \tag{1.1}$$

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Here $t > 0$ and $x \in R^2_+$, $(u^\varepsilon, v^\varepsilon)$ represent the fluid velocity field, $p^\varepsilon$ is a scalar pressure, and the initial data satisfies compatibility condition

$$\partial_x u_0 + \partial_y v_0 = 0, \quad u_0(x,0) = v_0(x,0) = 0.$$

The anisotropic Navier–Stokes equations are widely used in geophysical fluid dynamics as a mathematical model for water flow in lakes and oceans, and also in the study of Ekman boundary layers for rotating fluids, see for instance ([4,18]). These equations appear when the domain has very different horizontal and vertical dimensions, the turbulent viscosity coefficients may not be isotropic in this case. Formally, let $\varepsilon \to 0$, we get the following system:

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + v^\varepsilon \partial_y u^\varepsilon + \partial_x p^\varepsilon - \partial_{xx} u^\varepsilon = 0, \\ \partial_t v^\varepsilon + u^\varepsilon \partial_x v^\varepsilon + v^\varepsilon \partial_y v^\varepsilon + \partial_y p^\varepsilon - \partial_{xx} v^\varepsilon = 0, \\ \partial_x u^\varepsilon + \partial_y v^\varepsilon = 0, \\ v^\varepsilon(t,x,0) = 0, \\ (u^\varepsilon, v^\varepsilon)(0,x,y) = (u_0, v_0). \end{cases} \tag{1.2}$$

We observe that the boundary condition above is sufficient to solve (1.2).

The problem of vanishing viscosity limits for the classical incompressible Navier–Stokes equations is a classical issue. In the absence of physical boundaries, it has been proved that the Navier–Stokes equations indeed convergence to the Euler equations in various functional settings, see [1,9]. Moreover, the anisotropic Navier–Stokes equations with vanishing vertical viscosity in the absence of a physical boundary were also studied in [2].

However, in the presence of physical boundaries, this problem is a challenging problem due to the possible formation of boundary layers. For the Navier slip boundary condition, the boundary layer is weak and has been studied by many authors, see [25,24,7,8,15]. In particular, Rousset and Masmoudi introduce the conormal functional space to justify the limit from the Navier–Stokes to the Euler equation.

While, for the no-slip boundary condition, the boundary layer is strong and Prandtl formally derived the Prandtl equation in [19], which is a nonlinear degenerate parabolic–elliptic couple system. To justify this formal expansion, we must first establish the well-posedness of this equation. Up to now, well-posedness of Prandtl equation is only established in some special functional space. Under monotonic assumptions on the velocity of the outflow, Oleinik and her collaborators established the local existence of classical solutions in 2d. In this case, the global existence of weak solution was established for the favorable pressure by Xin and Zhang in [26]. Recently, Alexandre et al. and Masmoudi and Wong independently proved the local well-posedness in Sobolev space by the direct energy method. In [20], Sammartino and Caflisch obtained the local existence of analytical solution in the framework of the abstract Cauchy–Kowaleskaya theory, also see [12,4]. On the other hand, Gerard–Varet and Dormy proved the ill-posedness in Sobolev space for the linearized Prandtl equation around non-monotonic shear flows. We refer to [5] and reference therein for more relevant results.

However, there are only few results on the rigorous verification of the Prandtl boundary layer expansion. In [21], Sammartino and Caflisch achieved this in analytical setting and Wang et al. [23] give another proof by a direct energy method. Recently, Y. Maekawa given a rigorous verification of the Prandtl boundary layer expansion in half plane when the initial vorticity is supported

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