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Inertial manifolds for the hyperviscous Navier–Stokes equations

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Abstract

We prove the existence of inertial manifolds for the incompressible hyperviscous Navier–Stokes equations on the two or three-dimensional torus:

$$\begin{cases} u_t + \nu(-\Delta)^\beta u + (u\cdot\nabla)u + \nabla p = f, & (t,x)\in\mathbb{R}_+\times\mathbb{T}^d, \\ \operatorname{div} u = 0, \end{cases}$$

where $d = 2$ or 3 and $\beta \geq 3/2$. Since the spectral gap condition is not necessarily satisfied for the aforementioned problem in three dimensions, we employ the spatial averaging method introduced by Mallet-Paret and Sell in [26].

Keywords: Inertial manifold; Navier–Stokes equations; Global attractor; Hyperviscosity

1. Introduction

Let us consider the following hyperviscous version of the Navier–Stokes equations of incompressible fluid flow on a torus,

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$$\begin{cases} u_t + \nu(-\Delta)^\beta u + (u \cdot \nabla)u + \nabla p = f, & (t,x) \in \mathbb{R}_+ \times \mathbb{T}^d, \\ \operatorname{div} u = 0, \quad u|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where $\beta \geq d/4 + 1/2$ for the spatial dimension $d \geq 3$, and $\beta \geq 1$ for $d = 2$. Here, $\mathbb{T}^d = [-\pi, \pi]^d$ is endowed with the periodic boundary condition. It is well-known that the regularized system (1.1) is globally well-posed in the L^2 space (see, e.g., [8]). It has been used, among others, by numerical analysts as a substitute model for the standard case $\beta = 1$ and plays a key role in understanding turbulent phenomena in science (cf. Avrin [2], Borue and Orszag [4], Basdevant et al. [6], Browning and Kreiss [7], Cerruto et al. [10], Fornberg [20] and McWilliams [27]). Besides, the system (1.1) has some physical meaning (see again [10]). Given the nonlinear nature of turbulent incompressible viscous flows and the ensuing multiscale interactions, the direct numerical simulation of the Navier–Stokes equations is still presently lacking apart from some investigations performed on regularized systems which still retain the basic nonlinear structure and the essential features of the full hydrodynamic Navier–Stokes equations (cf. Holst et al. [22], Gal and Medjo [21]). Among such regularized models, it is worth noting the following globally well-posed systems in three dimensions: the Leray-α model (cf. Cheskidov et al. [11]), the simplified Bardina model (cf. Cao et al. [9]), the Navier–Stokes-α equations (cf. Foias et al. [16]) and the Navier–Stokes–Voight equations (cf. Kalantarov and Titi [23], Coti-Zelati and Gal [14]). One advantage of using the hyperviscous Navier–Stokes equations (1.1) with $\beta \geq 5/4$ in the three-dimensional domain is that the system only modifies the spectral distribution of energy, and the global well-posedness (i.e., existence, uniqueness and stability with respect to the initial data) of the solutions can be rigorously proven unlike for the three-dimensional Navier–Stokes equations (see [2,8]). On the other hand, (1.1) has also been proposed and investigated for the purpose of direct numerical simulations of turbulent incompressible viscous flows, although orders of dissipation $\beta \geq 2$ have typically been used (see again [7,10,20,27]). The existence of finite-dimensional global attractors for the regularized family (1.1) can be proven by employing standard theory of attractors. For the sake of completeness, we provide a brief proof of the global well-posedness and existence of global attractors of (1.1) in the appendix.

Furthermore, by directly verifying a spectral gap condition it was shown in [33] that the hyperviscous Navier–Stokes equations (1.1) possess an inertial manifold in L^2 provided $\beta > d$ for any spatial dimension $d \geq 2$. Exploiting the same strategy and a more refined analysis, the existence of inertial manifolds for (1.1) with $\beta > 5/2$ in three-dimensional domains was established in [3] (the results in [3] actually hold for a general family of hyperviscous operators). The purpose of this manuscript is to prove the existence of an inertial manifold for (1.1) with $\beta \geq 3/2$ in two or three dimensions when the spectral gap condition is not necessarily fulfilled. The motivation for this line of research is the long-standing open problem concerning the existence of inertial manifolds for the Navier–Stokes equation ($\beta = 1$).

We recall that the existence of an inertial manifold guarantees that, in the long-term, the system resembles a finite-dimensional system of ordinary differential equations, which describes the limit dynamics of the original system as time goes to infinity (see, e.g. [18,35]). Indeed, spectral gap conditions have been widely used in the literature to establish the existence of inertial manifolds for many dissipative evolution equations (cf. [1,13,17–19,32,34]). However, for a system that lacks the spectral gap condition, in their pioneering work [26], Mallet-Paret and Sell have introduced the so-called spatial averaging method to prove the existence of inertial manifolds for a three-dimensional reaction–diffusion equation. This technique was further simplified by Zelik [35], and extended by Kostianko and Zelik to the three-dimensional Cahn–Hilliard equation [25], and by Kostianko to the three-dimensional modified Leray-α model [24].

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