ARTICLE IN PRESS YJDEQ:9367

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Journal of Differential Equations

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The restricted three body problem on surfaces of constant curvature

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Received 23 November 2016; revised 25 April 2018

Abstract

We consider a symmetric restricted three-body problem on surfaces $\mathbb{M}^2_\kappa$ of constant Gaussian curvature $\kappa \neq 0$, which can be reduced to the cases $\kappa = \pm 1$. This problem consists in the analysis of the dynamics of an infinitesimal mass particle attracted by two primaries of identical masses describing elliptic relative equilibria of the two body problem on $\mathbb{M}^2_\kappa$, i.e., the primaries move on opposite sides of the same parallel of radius a. The Hamiltonian formulation of this problem is pointed out in intrinsic coordinates. The goal of this paper is to describe analytically, important aspects of the global dynamics in both cases $\kappa = \pm 1$ and determine the main differences with the classical Newtonian circular restricted three-body problem. In this sense, we describe the number of equilibria and its linear stability depending on its bifurcation parameter corresponding to the radial parameter a. After that, we prove the existence of families of periodic orbits and KAM 2-tori related to these orbits.

MSC: primary 70F07; secondary 70G60, 37D40

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https://doi.org/10.1016/j.jde.2018.06.007

0022-0396/

Please cite this article in press as: J. Andrade et al., The restricted three body problem on surfaces of constant curvature, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.06.007

Keywords: The restricted three body problem; Surfaces of constant curvature; Hamiltonian formulation; Periodic solutions; KAM tori; Averaging theory

1. Introduction

A first approach of the N-body problem on non Euclidean spaces was raised simultaneously by N. Lovachevsky [20], and by J. Bolyai [3] in the 1830's. In 1860 P.J. Serret extended the gravitational force to the sphere $\mathbb{S}^2$ and solved the corresponding Kepler problem [28]. In 1885, W. Killing adapted the Bolyai–Lobachevsky gravitational law to $\mathbb{S}^3$ and introduced the cotangent potential as a good extension of the classical potential [16].

One of the fundamental results on the area is due to H. Liebmann, who in 1902 proved that the orbits of the curved Kepler problem are indeed conics in $\mathbb{S}^3$ and $\mathbb{H}^3$ and a bit later, he proved an important result corresponding to an $\mathbb{S}^2$- and $\mathbb{H}^2$-analogues Bertrand's theorem, which states that there exist only two analytic potentials in the Euclidean space such that all bounded orbits are closed (see [18,19]).

Relevant and more recent works correspond to those obtained by V. Kozlov [17] and, particularly, contributions to the two-dimensional case of the Kepler problem belong to J. Cariñena, M. Rañada and M. Santander [4], who provided a unified formulation of the Kepler problem for an arbitrary constant curvature κ, getting conics orbits depending on the curvature parameter, showing that the well known conics orbits in Euclidean spaces can be extended to spaces of constant curvature. A similar approach of the two dimensional curved Kepler problem, but with emphasis on the dynamics, was treated in [1].

The curved N-body problem for $N \geq 2$ was introduced by F. Diacu et al. in a couple of papers *The N-body problem in spaces of constant curvature. Parts I and II* [11], [12], where a clear and detailed deduction of the equations of motion for any number $N \geq 2$ of bodies is presented, using variational methods of the theory of constrained Lagrangian dynamics, getting the equations of motion depending on the curvature κ. Some other interesting results in this direction can be found in [8], [9], [6], a complete historical background is given in [7].

We know that all 2-dimensional spaces of constant curvature κ are characterized by the sign of the curvature as the surfaces

$$\mathbb{M}^2_\kappa = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + \sigma z^2 = \kappa^{-1}\},$$

with $\sigma = sign(\kappa)$, which induces to define the inner product

$$(a_x, a_y, a_z) \odot (b_x, b_y, b_z) := a_x b_x + a_y b_y + \sigma a_z b_z.$$

If $\kappa > 0$, then the surface is the 2 dimensional sphere of radius $R = 1/\sqrt{\kappa}$ denoted by $\mathbb{S}^2_\kappa$ embedded in the Euclidean space $\mathbb{R}^3$. If $\kappa = 0$, we recover the Euclidean space $\mathbb{R}^2$. If $\kappa < 0$, then the surface is the upper sheet of the hyperboloid $x^2 + y^2 - z^2 = \kappa^{-1}$, with $z > 0$, denoted by $\mathbb{H}^2_\kappa$, which corresponds to a surface embedded in the 3 dimensional Minkowski space $\mathbb{R}^{2,1}$ ($\mathbb{R}^3$ endowed with the Lorentz inner product).

According to [11] the equations of motion for $N \geq 2$ bodies on a surface of curvature κ are given by