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An accurate and asymptotically compatible collocation scheme for nonlocal diffusion problems ☆

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ABSTRACT

In this paper, we develop and analyze a collocation scheme for solving the linear nonlocal diffusion problem with general kernels. To approximate the nonlocal diffusion operator, we take a classic trapezoidal rule based on the linear interpolation as the starting point, and then carefully derive a new improved quadrature rule, which is not only more accurate but also could avoid the evaluations of singular integrals. We then use this rule to construct a collocation scheme for solving the nonlocal diffusion equations, that produces a symmetric positive definite stiffness matrix with Toeplitz structure. The proposed scheme is rigorously shown to be of second order accurate with respect to the mesh size for the nonlocal problem with fixed horizon, and in particular, it can achieve higher order accuracy for the commonly used kernels in the literature. Furthermore, we also prove that the scheme is asymptotically compatible, i.e., the approximate solution of the nonlocal diffusion problem converges to the exact solution of the corresponding local PDE problem when the horizon and the mesh size both go to zero. Finally, numerical experiments are presented to verify the theoretical results.

1. Introduction

Nonlocal phenomenon are ubiquitous in nature and nonlocal models have appeared in many subjects, from physics and biology to materials and social sciences. For example, there has been a great deal of interest recently in the nonlocal peridynamics (PD) continuum theory introduced first by Silling [21–25]. PD model is an integral-type nonlocal continuum model for the mechanics of materials, which provides an alternative setup to that of classic continuum mechanics based on partial differential equations (PDEs). Linear scalar PD operators also share similarities with nonlocal diffusion operators, as pointed out in [8], thus making the study of PD relevant to the study of general nonlocal diffusion (ND) models in various applications [2–5,13–15,17,19]. Mathematical analysis of PD and ND models can be found in [5,7–9,11–13]. Various discretization methods have also been proposed for the spatial nonlocal PD models, including finite element [1,6,10,16,20,26,28,29,33], finite difference [26], collocation [30–32] and meshfree methods [18].

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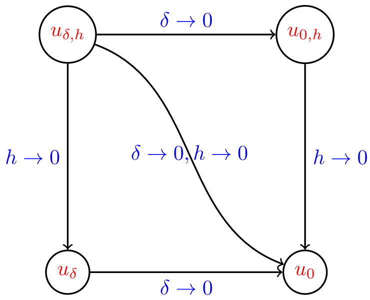

Fig. 1. A diagram for asymptotically compatible schemes and convergence results.

It is well-known that a common feature of PD and ND models is the introduction of the horizon parameter δ that characterizes the range of nonlocal interactions [9,21]. As $\delta \to 0$, the nonlocal effect diminishes and the zero-horizon limit of nonlocal PD models becomes a classical local PDE model when the latter is well-defined. Such limiting behavior provides connections and consistencies between nonlocal and local models, and has immense practical significance especially for multiscale modeling and simulations. A natural question is how such limiting behaviors can be preserved in various discrete approximations. This is a critical issue in the applications of PD like models to problems involving possibly different scales, given the popularity and practicality to perform PD simulations with a coupled horizon δ and mesh spacing h. Recently, for certain classes of parametrized problems, Du et al. [27] have introduced the concept of *asymptotically compatible* schemes and also established an abstract mathematical framework for their rigorous numerical analysis. For completeness, we here give an exposition of this concept and take the ND model as an example. Let u_δ and u_0 be the exact solution of the ND model and its limiting local PDE model, respectively, and let $u_{\delta,h}$ and $u_{0,h}$ be their corresponding discretization solutions. While we have the convergence of $\{u_{\delta,h}\}$ for a given δ as $h \to 0$, as well as the convergence of u_δ to u_0 and $u_{\delta,h}$ to $u_{0,h}$ as $\delta \to 0$, we are also interested in the behavior as both $\delta \to 0$ and $h \to 0$. So the following definition is given.

Definition 1. [27] A family of convergent approximations $\{u_{\delta,h}\}$ is said to be asymptotically compatible to the solution u_0 if as $\delta \to 0$ and $h \to 0$, we have $u_{\delta,h} \to u_0$.

It can be seen from Fig. 1 that an asymptotically compatible scheme can not only ensure to solve the ND model well if $\delta > 0$ but also obtain an approximation solution of the limiting local differential equation model as $\delta \to 0$. Therefore, the investigation of the asymptotically compatible schemes has directive function for the development of discretization schemes in the numerical simulation of ND models. The major contribution of this work is to develop an asymptotically compatible collocation scheme for solving the ND models.

As we pointed out before, the finite element discretization has been widely investigated for PD and ND models. But being based on variational formulations, it runs up against the embarrassment of the generation of stiffness matrices, and the difficulty of computing the entries become particular evident in higher dimension, due to the higher-order singularity of the inner and outer integrals as well as the irregular domains of integrations. An alternative way is to seek collocation discretization schemes due to their simplicity of implementation. In this work, we start with a classic trapezoidal rule based on the piecewise linear interpolation and then develop an improved quadrature rule for approximating the nonlocal operator, with their optimal error estimates being derived. This new rule also often could avoid the evaluations of singular integrals. Then based on this rule, we construct a collocation discretization scheme for solving nonlocal diffusion problems. By analyzing the stiffness matrices we show that the proposed scheme produces a symmetric positive definite discrete system with Toeplitz structure and satisfies the discrete maximum principle, and then starting from this, the following conclusions are further rigorously obtained: (1) the scheme is of second order convergent for the nonlocal diffusion problem with general kernels when the horizon is fixed; (2) if the kernel of the nonlocal operator is specified by a commonly used form in the literature, the convergence order is even higher (up to 3), which shows superiority compared to the quadrature-based finite difference method developed in [26]; (3) the scheme is also asymptotically compatible.

The rest of the paper is organized as follows. In Section 2, we first describe the nonlocal diffusion problem and give some useful notations. Two quadrature rules for approximating the nonlocal operator, including their formulation and error estimates, are discussed in Section 3. In Section 4, a collocation scheme is proposed for solving the nonlocal diffusion problem, along with some properties of the discretized system be discussed. In Section 5, the convergence and error estimates of the proposed scheme are proved, and we also show that the scheme is asymptotically compatible. Results of numerical experiments are reported in Section 6 to complement the theoretical analysis. Some conclusions are given in Section 7.

2. Mathematical model and some notations

Let Ω be a finite bar in $\mathbb{R}$. Without loss of generality, we take $\Omega = (0, 1)$. Let $\delta > 0$ be a horizon parameter. A nonlocal diffusion operator $\mathcal{L}_\delta$ is defined as, for any function $u = u(x) : \Omega \to \mathbb{R}$,

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