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On a parallel, 3-dimensional, finite element solver for viscous, resistive, stationary magnetohydrodynamics equations: Velocity–current formulation

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ABSTRACT

We describe a parallel implementation for the numerical approximation of solutions to the three-dimensional viscous, resistive magnetohydrodynamics (MHD) equations using a velocity–current formulation. In comparison to other formulations, the velocity–current formulation presented in this paper is an integro-differential system of equations that incorporates nonideal boundaries and nonlinearities due to induction. The solution to the equations is approximated using a Picard iteration, discretized with the finite element method, and solved iteratively with the Krylov subspace method GMRES. Effective preconditioning strategies are required to numerically solve the resulting equations with Krylov solvers [12]. For GMRES convergence, the system matrix resulting from the discretization of the velocity–current formulation is preconditioned using a simple, block-diagonal Schur-complement preconditioner based on [14]. The MHD solver is implemented using freely available, well-documented, open-source, libraries deal.II, p4est, Trilinos, and PETSc, capable of scaling to tens of thousands of processors on state-of-the-art HPC architectures.

1. Introduction

MHD (magnetohydrodynamics) is the study of electrically conducting fluids in the presence of electromagnetic fields. MHD phenomena span from the earth's magnetosphere, solar flares, collapse and formation of stars and galaxies [35,13,24,25] to devices for drug delivery and image capturing in the human body by way of MHD micropumps and MHD propulsion [32,21,41]. Of particular interest are applications modeled by the visco-resistive MHD equations: liquid metal flows in aluminun casting, the Czochralski crystal growth process for silicon used in the semiconductor industry, and fusion plasmas for controlled fusion energy [29,19,15,18,1,12,38].

We describe a numerical method for simulating and visualizing three-dimensional magnetohydrodynamics flows. The method utilizes the velocity–current formulation of the viscous, resistive MHD equations (proposed in [26]). In comparison

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to other formulations, the velocity–current formulation is an integro-differential system of equations that accounts for non-ideal boundaries and nonlinearities due to induction [28]. The solution to the velocity–current formulation is approximated using a Picard linearization. The Picard linearization allows for a sparse system matrix [28]. The equations are discretized using the finite element method. The implementation of the method uses the core library deal.ii, an open-source, object-oriented library for the rapid development of adaptive finite element methods, utilizing multithreading, MPI and wrapping to support libraries for parallelization [2]. The support library p4est was used to partition and distribute the discretized formulation among the nodes of the distributed memory cluster. p4est enables adaptive mesh refinement (AMR) on a collection (forests) of octrees (scaling to hundreds of thousands of processor cores) [3,9].

Effective preconditioning strategies are required to numerically solve the resistive MHD equations [12]. The discretization of the velocity–current formulation results in a system matrix similar to that of the Navier–Stokes equations, and work on Schur-complement preconditioners for the Navier–Stokes equations [14] is utilized. Using deal.ii to wrap into Trilinos' solver framework [20], we construct a preconditioner to iteratively solve the discretized system. The preconditioner is a simple, efficient block-diagonal preconditioner. We demonstrate convergence of the proposed method on two test problems. The approximate solutions were visualized using VisIt, an open source, interactive, parallel visualization and analysis tool for data defined on two and three-dimensional meshes [11].

The paper is organized as follows. Section 2 introduces the viscous, resistive MHD equations using the velocity–current formulation. Section 3 discusses the weak formulation and the LBB (Ladyzhenskaya, Babuska, Brezzi) condition, used to determine finite element pairings and construct the matrix system preconditioner. Sections 4 and 5 discuss the linear system and the block-diagonal preconditioner. Sections 6 and 7 provide a code flowchart and description of the numerical simulations which illustrate attainable convergence rates, strong scaling, and weak scaling, and concluding remarks.

2. Resistive MHD formulation

The viscous, resistive MHD equations using the velocity–current formulation (below) model the interaction of a stationary (steady-state), conductive, incompressible fluid flow (contained within a domain Ω) with electrical and magnetic fields both within and outside the domain. Employing the electric current density rather than the magnetic field as the primary electromagnetic variable, it is possible to avoid artificial or highly idealized boundary conditions for the electric and magnetic fields (where electromagnetic interactions outside the domain are neglected) and to account exactly for the electromagnetic interaction of the fluid with the surrounding media [29,36]. The velocity–current formulation also allows easy incorporation of contributions from experimentally measurable external currents when these are present.

The formulation consists of the stationary Navier–Stokes equations, Ohm's Law, the continuity equations (conservation of mass and current), and Maxwell's equations. The first four equations are:

$$-\eta\Delta\mathbf{u} + \rho\,(\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla p - \mathbf{J}\times\mathbf{B} = \mathbf{F} \qquad \text{in } \Omega\,, \tag{1}$$

$$\sigma^{-1}\mathbf{J} + \nabla\phi - \mathbf{u}\times\mathbf{B} = \mathbf{E} \qquad \text{in } \Omega\,, \tag{2}$$

and

$$\nabla\cdot\mathbf{u} = 0 \quad \text{and} \quad \nabla\cdot\mathbf{J} = 0 \qquad \text{in } \Omega\,. \tag{3}$$

In the momentum equation (1) the viscosity η and density ρ are assumed to be constant. The unknowns are the velocity $\mathbf{u}$, the pressure p, the current density $\mathbf{J}$, and the magnetic field $\mathbf{B}$. The body force $\mathbf{F}$ is assumed to be given. The flow interacts with electric currents and magnetic fields by way of the Lorentz force, $\mathbf{J}\times\mathbf{B}$, term in the equation. Coupled to the Navier–Stokes equations by way of the magnetic induction, $\mathbf{u}\times\mathbf{B}$, (and the Lorentz force) is Ohm's law (2), a constitutive equation. An additional unknown in Ohm's law is the electric potential ϕ. The fluid conductivity σ is assumed to be constant and an additional electric field $\mathbf{E}$ may be given (or prescribed, however in most physical situations it is zero). The continuity equations (3) are a consequence of fluid incompressibility, conservation of mass, and conservation of charge.

Equations (1)–(3) can be simplified by eliminating the magnetic field $\mathbf{B}$ using the Biot–Savart law:

$$\begin{aligned}\mathbf{B}(\mathbf{x}) &= \mathbf{B}_0(\mathbf{x}) + \mathcal{B}(\mathbf{J})(\mathbf{x})\\ &= \mathbf{B}_0(\mathbf{x}) - \frac{\mu}{4\pi}\int_{\Omega}\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\times\mathbf{J}(\mathbf{y})\,d\mathbf{y}\end{aligned} \tag{4}$$

where $\mathbf{B}_0$ is the sum of fields due to given external sources, $\mathcal{B}(\mathbf{J})$ is the field generated by induction, and μ is the magnetic permeability of the fluid, which is also assumed to be constant. Similarly, the magnetic field due to external sources is

$$\mathbf{B}_0(\mathbf{x}) = \mathbf{B}_{ext}(\mathbf{x}) - \frac{\mu}{4\pi}\int_{\mathbb{R}^3\setminus\overline{\Omega}}\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\times\mathbf{J}_{ext}(\mathbf{y})\,d\mathbf{y}\,, \tag{5}$$

where $\mathbf{B}_{ext}$ accounts for possible external magnetic fields other than ones induced by known currents $\mathbf{J}_{ext}$ outside the domain. Additional constraints are the conservation of charge $\nabla\cdot\mathbf{J}_{ext} = 0$ for the external current(s) and Gauss's law for the magnetic field $\nabla\cdot\mathbf{B}_{ext} = 0$ in $\mathbb{R}^3$.