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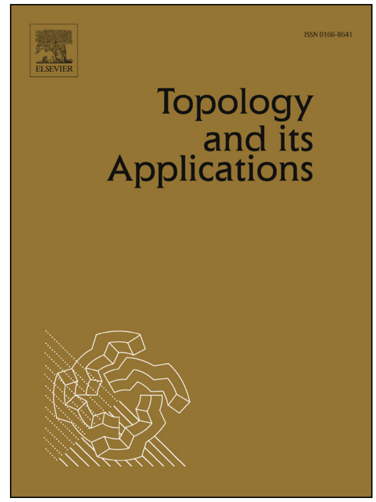

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ON ISOMETRIES OF SYMMETRIC PRODUCTS OF METRIC SPACES

NAOTSUGU CHINEN

Abstract. By $F_n(X)$, $n \geq 1$, we denote the n-th symmetric product of a metric space (X, d) as the space of the nonempty finite subsets of X with at most n elements endowed with the Hausdorff metric d_H. By $\mathrm{Iso}(X)$ we denote the group of all isometries from X onto itself with the topology of pointwise convergence. In this paper, we show that, under the certain hypothesis, $\mathrm{Iso}(F_n(X))$ is topologically isomorphic to the semidirect product group $\mathrm{Iso}(F_n(X), F_1(X)) \rtimes \mathrm{Iso}(X)$. We apply those results to ℓ_p^q, $(p, q) \in [1, \infty] \times \mathbb{N}^*_{\geq 2}$, as particular spaces and prove the following statements:

(1) If $p \in \{1, \infty\}$, then $\mathrm{Iso}(F_2(\ell_p^2))$ is topologically isomorphic to $\mathbb{Z}_2 \times \mathrm{Iso}(\ell_p^2)$.
(2) If $3 \leq q < \infty$, then $\mathrm{Iso}(F_2(\ell_\infty^q))$ is topologically isomorphic to $\prod_{i=1}^{q-1} (\mathbb{Z}_2)_i \rtimes \mathrm{Iso}(\ell_\infty^q)$.
(3) In other cases except $(n, p, q) \in \mathbb{N}_{\geq 2} \times \{1, \infty\} \times \{\infty\}$, the canonical homomorphism $\chi_n : \mathrm{Iso}(\ell_p^q) \to \mathrm{Iso}(F_n(\ell_p^q))$ is a topological isomorphism.

1. Introduction

As an interesting construction in topology, Borsuk and Ulam [4] introduced the n-th *symmetric product* of a metric space (X, d), denoted by $F_n(X)$. Recall that $F_n(X)$ is the space of nonempty finite subsets of X with at most n elements endowed with the Hausdorff metric d_H (see [12, p.6] or Definition 2.2 below). A considerable number of studies have been made on the topological structures of $F_n(X)$ (see [8] and [15]).

For a metric space (X, d), we denote by $\mathrm{Iso}(X)$ the group of all isometries from (X, d) onto itself with the topology of pointwise convergence (see [1, p.173]). Let $n \in \mathbb{N}$. Every isometry $\phi : (X, d) \to (X, d)$ induces an isometry $\chi_n(\phi) : (F_n(X), d_H) \to (F_n(X), d_H)$ defined by $\chi_n(\phi)(A) = \phi(A)$ for each $A \in F_n(X)$. Thus, we can define the natural monomorphism $\chi_n : \mathrm{Iso}(X) \to \mathrm{Iso}(F_n(X))$. We can easily see that this monomorphism is a topological embedding. Therefore, it follows that

$(\star)_n$ $\chi_n : \mathrm{Iso}(X) \to \mathrm{Iso}(F_n(X))$ is a topological isomorphism (i.e., a group isomorphism and a homeomorphism)

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