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# Linear fuzzy game with coalition interaction and its coincident solutions

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**Abstract**

In this study, we consider the solution concepts for fuzzy coalition games (i.e., cooperative games with fuzzy coalition) under a certain participation level. In general, cooperative games with fuzzy coalition are based on the assumption that all the fuzzy coalition values for different fuzzy coalitions must be represented by the same formula, which may omit the coalition interaction under different participation ratios for players. Considering these conditions, we propose the coincident fuzziness form for games with fuzzy coalition, which are represented by a mapping from the characteristic function of the crisp game to that of the fuzzy coalition game. The proposed fuzzy coalition games admit the differences in coalition interactions for different fuzzy coalitions, where the coalition interactions are represented by the fuzzy coalition values in different ways (or formulas). For a fixed fuzzy coalition, the maximum fuzzy coalition game is proven to be the Choquet integral form on the condition that the associate crisp game is convex. In order to seek appropriate solutions for the proposed games based on a certain participation level, the *Fuzzy*-Shapley axioms are defined, and the explicit Shapley value is represented by the Shapley value of the associated crisp games. Moreover, the fuzzy core of this proposed fuzzy coalition game is the stable solution set for the given fuzzy coalition, which is also denoted by the crisp cores. Furthermore, we study the relationship between the *Fuzzy*-Shapley value and the fuzzy core. By adding restrictions on the fuzzy core, we propose a strong *Fuzzy*-core, which is a more stable solution than the fuzzy core. In addition, the fuzzy core is equivalent to the strong *Fuzzy*-core under some condition.


*Keywords:* Cooperative game; Core; Fuzzy coalition; Shapley value

## 1. Introduction

The classical solutions in the crisp cooperative games assume that all players are peers, which depends on the set of cooperating players and the mapping of the game. In real life, social or economic circumstances may impose certain

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restraints on the formation of coalitions according to the relationships or resource constraints among the players. Thus, several studies have developed models of cooperative games with partial cooperation.

For fuzzy coalitions, Aubin [1,2] first introduced a game where some players only participate partially. Butnariu [3–5] defined the games with proportional values and showed the explicit form of the Shapley function based on that class of fuzzy games. Tsurumi et al. [6] defined a new class of fuzzy games with the Choquet integral form and another explicit form of the Shapley function. Li et al. [7] showed that the explicit form of the Shapley value introduced by Butnariu and Tsurumi et al. could be written using the same expression. Borkotokey [8] studied fuzzy games where the coalition and the characteristic function are both fuzzy information. Jimenez-Losada [9] investigated a fuzzy coalition cooperative game and its Shapley value based on a partition function. Many researchers have considered other extensions of fuzzy coalition games in order to discuss the fuzzy extension of the Shapley value. For example, Meng et al. [10] introduced the fuzzy Owen value as a solution for some major types of fuzzy coalition games in coalition structure form, such as games with proportional value, fuzzy games with Choquet integral form, and multilinear extension games [11]. Liao [12] considered a fuzzy extension of the Shapley value for a fuzzy coalition game, which they called the consistent value. Borkotokey et al. [13] introduced fuzzy bi-cooperative games in multilinear extension form and obtained the value solution.

In addition to the Shapley value, other solutions have been studied for fuzzy coalition games. The lexicographical solution for fuzzy games was considered by Sakawa and Nishizaki [14]. The core for fuzzy games was also studied by Tijs et al. [15]. Yu et al. [16] defined the fuzzy core for any type of fuzzy coalition game and gave the explicit form of the fuzzy cores in crisp cores for the multilinear extension introduced by Owen [11], the games with proportional value proposed by Butnariu, and the games with Choquet integral form given by Tsurumi et al. Wu [17] proposed many types of proper cores and dominance cores for fuzzy games, as well as analyzing the relationship between all types of cores. Mielcová [18] discussed the construction of the core of a transferable utility cooperative game when the possible coalitions of agents are vague. Sagara [19] discussed the cores and Weber sets for extensions of cooperative games, and also analyzed the coincidence of the solutions for the fuzzy and crisp games.

All of the different types of cooperative fuzzy coalition games mentioned above are based on an assumption that all the fuzzy coalition values for the different fuzzy coalitions must be represented by the same formula, i.e., the multilinear extension introduced by Owen, the games with proportional value proposed by Butnariu, or the games with Choquet integral form given by Tsurumi et al. However, we can see that the fuzziness of fuzzy coalition game "$v$" should be obtained from the crisp cooperative game, so the fuzzy coalition values for different fuzzy coalitions do not need to be represented by the same formula (see Example 3.1). Thus, if different fuzzy coalition values are extended in the same manner by crisp cooperative games, then we may omit the coalition interaction under different participation ratios for players. In fact, if and only if the interaction role of the big crisp coalition "$N$" always exists for any degree of participation by players, then the fuzziness of the fuzzy coalition value can be denoted by the same formula (see Example 3.2). However, in a crisp cooperative game, the players in a coalition "$N$" always wander to participate in the big coalition "$N$" under a certain condition, e.g., that the resources of all the players are totally invested in the big coalition "$N$" and the invested capital is sufficient for the fixed assets. Hence, if the fuzzy coalition is formed under the restriction of partly investing resources, then the resource advantage of crisp coalition "$N$" may disappear. In this situation, the leader of the players may differ, so the fuzzy coalition value may not be defined based on the assumption that the interactive role is always the same with the big coalition "$N$" (see Example 3.2).

In this study, we aim to define a new class of fuzzy coalition games, which admits the differences in interactions for different fuzzy coalitions. This class of fuzzy coalition games denotes different fuzzy coalition values in different ways (or formulae). In order to seek appropriate solutions for the proposed class game based on a certain fuzzy coalition "$U$," we also study the *Fuzzy*-Shapley value and fuzzy cores for this new class of games with fuzzy coalition. It should be noted that the class of fuzzy coalition games contains several types of fuzzy games, including the multilinear extension, games with proportional value, and games with the Choquet integral form. Hence, the new class of fuzzy coalition games introduced in this study can be considered a generalized form of fuzzy games. The *Fuzzy*-Shapley value and fuzzy core for this class of games with fuzzy coalition are expressed by the crisp Shapley value and crisp cores, respectively.

The remainder of this paper is organized as follows. In Section 2, we review some definitions of crisp cooperative game and games with fuzzy coalition. In Section 3, we proposed a new class of fuzzy coalition games called linear fuzzy games. We investigate the relationship between the *Fuzzy*-Shapley value and the crisp ones. In Section 4, we consider the fuzzy core of the linear fuzzy game and its relationship with *Fuzzy*-Shapley values. The strong *Fuzzy*-core

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