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Nontransferable utility games with fuzzy coalition restrictions

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Abstract

A value for nontransferable utility (NTU) games with fuzzy coalition restrictions is introduced and characterized. In a similar way as the Shapley value for transferable utility (TU) games has been extended in the literature to study games with restricted cooperation, we extend the Shapley NTU value to deal with NTU games in situations in which there are fuzzy dependency relationships among the players.

Keywords: Nontransferable utility games; Shapley NTU value; Restricted cooperation; Fuzzy coalition

1. Introduction

Cooperative game theory deals with groups of players that aim to share the benefits derived from their cooperation. The cooperative model abstracts away from some details of the interaction among the players and describes only the outcomes that result when players cooperate within different coalitions. At this point, two different approaches can be considered. In some situations, the outcome of each coalition is described by a real number. The games used in these cases are called transferable utility (TU) games. The adjective transferable refers to the assumption that a player can losslessly transfer any part of his utility to another player, usually through money, and that the players' utilities are linear in money with the same scale for all players. If there is no possibility of transferring the utility between players by using money or, if there is, the scales between utilities and money are different, then nontransferable utility (NTU) games are used.

Given a cooperative game, it is often assumed that the players are free to participate in any coalition, but in some situations there are dependency relationships among the players that restrict their capacity to cooperate within some coalitions. Those relationships must be taken into account if we want to distribute the profits fairly. In this regard, different kinds of limitations on cooperation among players have been studied in the literature, and various structures have been used, such as the permission structures introduced by Gilles, Owen and van den Brink [5]. In Gallardo et al. [4] one of these models for games with restricted cooperation was introduced. This model is more general than others

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in several ways. For instance, it allows to deal with non-hierarchical or non-transitive dependency relationships. But the most important advantage of this model is that it allows to deal with fuzzy dependency relationships, which arise in situations in which each player has a degree of freedom to cooperate within a coalition. Our goal is to extend this model to NTU games. To this end, we will use the value for NTU games introduced by Shapley [7].

The paper is organized as follows. In Section 2, several basic definitions and results concerning cooperative games are recalled. In Section 3, fuzzy authorization structures are introduced. They will be used to model situations in which some players depend partially on other players. In Section 4, NTU games with fuzzy authorization structure are defined, and the Shapley fuzzy authorization NTU correspondence is defined and characterized. Finally, in Section 5, we draw some conclusions.

2. Preliminaries

2.1. Transferable utility games

A *transferable utility game* or *TU game* is a pair (N, v), where N is a set of cardinality n with $n \in \mathbb{N}$ and $v : 2^N \to \mathbb{R}$ is a function satisfying that $v(\emptyset) = 0$. The elements of N are called players, the subsets $E \subseteq N$ are called coalitions and the number $v(E)$ is the worth of E. Often, the TU game (N, v) is identified with the function v. The set of all TU games on N is denoted by $\mathcal{G}^N$.

Given a TU game (N, v), a problem that arises is how to assign a payoff to each player in a fair way. An allocation rule or value assigns to each game (N, v) a payoff vector $\psi(v) \in \mathbb{R}^N$. Many allocation rules have been defined in literature. The best known of them is the Shapley value, introduced by Shapley [6] in 1953. Given $v \in \mathcal{G}^N$, the *Shapley value of* v, denoted by $\phi(v)$, is defined as

$$\phi_i(v) = \sum_{\{E \subseteq N : i \in E\}} p_E \left[v(E) - v(E \setminus \{i\}) \right] \quad \text{for all } i \in N,$$

where $p_E = \dfrac{(n - |E|)!\,(|E| - 1)!}{n!}$ and $|E|$ denotes the cardinality of E.

2.2. Nontransferable utility games

A *cooperative game with nontransferable utility* or NTU game is a pair (N, V) where N is a set of cardinality $n \in \mathbb{N}$ and V is a correspondence that assigns to each nonempty $E \subseteq N$ a nonempty subset $V(E) \subseteq \mathbb{R}^E$. The set-valued function V is the characteristic function of the NTU game (N, V). Often, the NTU game (N, V) is identified with the function V.

If V and W are NTU games, the NTU game $V + W$ is defined by

$$(V + W)(E) = V(E) + W(E) = \{x + y : x \in V(E), y \in W(E)\}$$

for every $E \in 2^N \setminus \{\emptyset\}$.

If $\alpha \in \mathbb{R}^N$, the NTU game $\alpha * V$ is defined by

$$(\alpha * V)(E) = \left\{ \alpha^E * x : x \in V(E) \right\}$$

for every $E \in 2^N \setminus \{\emptyset\}$, where α^E is the restriction of α to E (i.e., $\alpha^E \in \mathbb{R}^E$ and $\alpha_i^E = \alpha_i$ for every $i \in E$) and $*$ denotes the Hadamard product (i.e., if $x, y \in \mathbb{R}^E$, then $x * y$ is the element in $\mathbb{R}^E$ defined as $(x * y)_i = x_i y_i$ for every $i \in E$).

Given $v \in \mathcal{G}^N$ the NTU game corresponding to v is defined as

$$V_v(E) = \left\{ x \in \mathbb{R}^E : \sum_{k \in E} x_k \leqslant v(E) \right\} \quad \text{for every nonempty } E \subseteq N.$$

Suppose C is a convex set in $\mathbb{R}^N$. We denote by ∂C the boundary of C and by $\overline{C} = C \cup \partial C$ the closure of C. Set C is comprehensive if $y \leqslant x$ and $x \in C$ imply $y \in C$. We recall that C is smooth if it has a unique supporting hyperplane at each point of ∂C.