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Short communication

A new view on Arrovian dictatorship in a fuzzy setting

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Abstract

In this note, we obtain new Arrovian results in the setting where the preferences of individuals are fuzzy and no properties of asymmetry are required. Dealing with a society made of possibly infinitely many individuals, we obtain the existence of a unique ultrafilter of decisive coalitions for every *Fuzzy Social Welfare Function* that obeys to Arrow's like axioms. Then, we deduce new impossibility and possibility results.

Keywords: Fuzzy preferences; Aggregation rules; Infinitely many agents; Arrovian dictatorship

1. Introduction

Starting from the seminal work by Arrow [1], the dictatorship in aggregation rules is a widely discussed issue in the literature. Several papers face the case where the individual preference relations are given by means of fuzzy binary relations. Among the others, Barrett et al. [3], Dutta [6], Banerjee [2], Richardson [14], Mordeson et al. [12], Gibilisco et al. [9], Fotso and Fono [8], Mordeson et al. [13] have been the departure point for the considerations we present in this article. In all the above papers, the societies count finitely many individuals.

In this note we study societies with not necessarily finitely many agents. Individuals rank alternatives by using fuzzy preferences, that we assume to be T_0*-transitive* and *negatively transitive*, without requiring asymmetry (see Section 2.1). So, this can be considered as a new setting where to investigate Arrow's like results. Indeed, Barrett et al. [3] consider fuzzy binary relations which are, in a fuzzy sense, irreflexive, asymmetric and T_0-transitive, while Dutta [6], Banerjee [2], Richardson [14], Mordeson et al. [12], Gibilisco et al. [9] and Fotso and Fono [8] deal with preferences which involve asymmetric fuzzy binary relations. Several examples point out that our setting is independent of those given in the mentioned literature. Despite of the lack of asymmetry as a primitive requirement on preferences, we give a connection between T_0-transitive and negatively transitive fuzzy binary relations and asymmetric and negatively transitive exact (we also say crisp) binary relations. We remark that, differently from the crisp case, where any asymmetric and negatively transitive binary relation is transitive, T_0-transitivity neither implies nor is implied by negative transitivity (see Examples 1 and 2) and both of them are independent from fuzzy asymmetry (see Remark 1).

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The connection mentioned above consists of an equivalence relation we introduce in the set of T_0-transitive and negatively transitive fuzzy binary relations (see Section 2.2). We show that every asymmetric and negatively transitive crisp binary relation can be identified with a unique equivalence class: see Proposition 2. So, our setting seems to appear as the natural fuzzy extension of the classical domain of crisp Arrow's like results. Moreover, the equivalence enables us to use the results of the "crisp literature" in order to obtain new fuzzy Arrovian results. In particular, we extend the Theorem of Kirman and Sondermann [10] to the fuzzy setting (see Theorem 1, our main result): every *Fuzzy Social Welfare Function* that satisfies Arrow's type axioms, if the underlying society is not necessarily finite, is determined by a unique ultrafilter of decisive coalitions. Then, we deduce from it a new Arrow's impossibility theorem in finite societies. We also present a Fishburn's possibility theorem [7] in the infinite case. See Section 3.

2. The setting

Let X be a non-empty set. We recall that a *fuzzy set* A in X is given by means of a *membership function* $\mu_A : X \longrightarrow [0, 1]$, where $\mu_A(x)$ represents the degree of membership of the element x in the fuzzy set A [15]. A *fuzzy binary relation* on X is a fuzzy set in $X \times X$. So, if R is a fuzzy binary relation on X and $f_R : X \times X \longrightarrow [0, 1]$ is the membership function of R, the value $f_R(x, y)$ is the degree to which x is preferred to y (from now on, we identify a fuzzy binary relation with its membership function).

In this section, we introduce our setting of fuzzy binary relations and fuzzy Arrow's type axioms where Arrovian results hold, as we shall show later.

2.1. Fuzzy preferences

In the following definition, we fix some properties for fuzzy binary relations (see Basu [4], Barrett et al. [3], Dutta [6]). Throughout the paper, we assume that X has at least three alternatives.

Definition 1. A fuzzy binary relation f on X is said to be:

i) *T_0-transitive* if, for all distinct x, y and z, $f(x, z) > 0$ and $f(z, y) > 0 \Longrightarrow f(x, y) > 0$;
ii) *negatively transitive* if, for all distinct x, y and z, $f(x, z) = 0$ and $f(z, y) = 0 \Longrightarrow f(x, y) = 0$.

The properties introduced in the above definition generalize to the fuzzy setting, respectively, the well known *transitivity* and *negative transitivity* of crisp binary relations.[1] Let us point out that T_0-transitivity neither implies nor is implied by negative transitivity, as the examples below show.

Example 1. Assume that $X = \{x, y, z\}$ has cardinality three, and let f be the fuzzy binary relation on X defined by:

f	x	y	z
x	0	1	0
y	1/2	0	1
z	0	1	0

It is easy to see that f is negatively transitive but not T_0-transitive.

Example 2. Let X be of cardinality four and let g be defined by:

g	x	y	z	w
x	0	0	0	1
y	1	0	0	1
z	0	0	0	0
w	1/2	0	0	0

Then, g is T_0-transitive but not negatively transitive.

[1] The usual (crisp) binary relations are the fuzzy ones with membership functions taking values in $\{0, 1\}$. We recall that a crisp binary relation $\succ$ is: *asymmetric* if, for all $x \neq y$, $x \succ y \Longrightarrow y \not\succ x$; *negatively transitive* if, for all distinct x, y and z, $x \not\succ z$ and $z \not\succ y \Longrightarrow x \not\succ y$.