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Well-posedness and decay of solutions for three-dimensional generalized Navier–Stokes equations

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ABSTRACT

In this paper, we study the generalized incompressible Navier–Stokes equations in $\mathbf{R}^3$. Based on the energy estimates and regularization of the initial data with the heat semi-group, we prove the well-posedness of solutions in $H^{\frac{5-4\alpha}{2}}(\mathbf{R}^3)$ provided that the $H^{\frac{5-4\alpha}{2}}(\mathbf{R}^3)$-norm of initial data is sufficiently small. In addition, in contrast to the generalized heat equation, the upper bound of the time decay rate of solutions to the generalized Navier–Stokes equations is also established.

1. Introduction

Consider the following generalized incompressible Navier–Stokes equations:

$$\begin{cases} u_t + u \cdot \nabla u + (-\Delta)^{\alpha} u + \nabla \pi = 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1}$$

where $u = (u_1, u_2, u_3) \in \mathbf{R}^3$ is the velocity field of the fluid and $\pi \in \mathbf{R}$ is the pressure, respectively. The fractional Laplacian operator $(-\Delta)^{\delta}$ is defined through the Fourier transform [1], namely

$$\widehat{(-\Delta)^{\delta} f}(\xi) = \widehat{\Lambda^{\delta} f}(\xi) = |\xi|^{2\delta} \widehat{f}(\xi),$$

and $\widehat{f}$ is the Fourier transform of f. Sometimes we write $\Lambda = (-\Delta)^{\frac{1}{2}}$ for notational convenience.

This system is of interest for various reasons. For example, it includes the well-known equations, say Navier–Stokes equations ($\alpha = 1$) [2–9]. In addition, it also has similar scaling properties and energy estimate as the Navier–Stokes equations.

In [10], Lions proved that when $\alpha \geq \frac{5}{4}$, the 3D generalized Navier–Stokes equations have a global and unique regular solution (see also in [11] and [12], the MHD equations reduce to the Navier–Stokes equations as the magnetic field $b = 0$). However, for the case $\alpha < \frac{5}{4}$, the global well-posedness theories of 3D generalized Navier–Stokes equations remain open. Consequently, considerable works are devoted to concerning the regularity criteria for this range (see for example [13,12,14,15]). Another direction is to obtain its existence of global solutions for system (1) with small initial data u_0 belonging to a variety of spaces, for example, the Besov spaces $\dot{B}^r_{2,1}$ and $B^r_{2,q}$ [16], the critical spaces $G_3^{-(2\alpha-1)}(\mathbf{R}^3)$ and $BMO^{-(2\alpha-1)}(\mathbf{R}^3)$ [17], the critical space $Q^{\beta,-1}_{\alpha;\infty}(\mathbf{R}^3)$ [18], the local Q-type spaces [19], the largest critical spaces $\dot{B}^{-2\alpha-1}_{\infty,\infty}(\mathbf{R}^n)$ [20], the critical Triebel–Lizorkin type oscillation spaces $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbf{R}^3)$ [21].

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Define the fractional-order (homogeneous) Sobolev spaces $\dot{H}^s(\mathbf{R}^3)$ by using Fourier transform:

$$\dot{H}^s(\mathbf{R}^3) = \{u \in \mathscr{P}' : \ \hat{u} \in L^1_{loc}(\mathbf{R}^3), \ \ |\xi|^s\hat{u}(\xi) \in L^2(\mathbf{R}^3)\},$$

with

$$\|u\|^2_{\dot{H}^s(\mathbf{R}^3)} := \int_{\mathbf{R}^3} |\xi|^{2s}|\hat{u}(\xi)|^2 d\xi,$$

where $\mathscr{P}'$ is the collection of all tempered distributions.

We want to prove the local existence for all initial data in the critical spaces $\dot{H}^{\frac{5-4\alpha}{2}}$ and global existence for initial data whose $\dot{H}^{\frac{5-4\alpha}{2}}$ norm is small enough. More precisely, the result can be stated as follows.

Theorem 1. *Suppose that* $\alpha \in (0, \frac{5}{4})$ *and* $u_0 \in H^{\frac{5-4\alpha}{2}}(\mathbf{R}^3)$. *Then*

- *there exists a time* $T = T(u_0) > 0$ *such that system (1) have a unique solution*

$$u(x, t) \in L^\infty(0, T; \dot{H}^{\frac{5-4\alpha}{2}}) \bigcap L^2(0, T; \dot{H}^{\frac{5-2\alpha}{2}}) \tag{2}$$

- *there is an absolute constant* ϵ *which is independent of* u_0 *such that if* $\|u_0\|_{\dot{H}^{\frac{5-4\alpha}{2}}} < \epsilon$, *then the solution satisfies (2) for every* $T > 0$ *and hence is unique for all* $t > 0$.

Remark 2. Since the 3D generalized Navier–Stokes equations have a global and unique regular solution when $\alpha \geq \frac{5}{4}$, we only consider the case $0 < \alpha < \frac{5}{4}$ in this paper.

Remark 3. The same strategy was developed in [9,22] for 3D Navier–Stokes equations. Theorem 1 can be viewed as complementary result of [9,22].

It was Jiu and Yu [23] who first studied the decay of solutions to the 3D generalized Navier–Stokes equations. Supposed that $0 < \alpha < \frac{5}{4}$ and $u_0 \in L^2(\mathbf{R}^3) \bigcap L^p(\mathbf{R}^3)$ with $\max\{1, \frac{1}{3-2\alpha}\} \leq p < 2$, the authors showed that the decay of the solution is

$$\|u\|^2_{L^2} \leq C(1+t)^{-\frac{3}{2\alpha}\left(\frac{2}{p}-1\right)}. \tag{3}$$

Analyze the above results, we find that there is one interesting problem need to be considered: Jiu and Yu [23] did not study the case $\alpha > 1$ and $u_0 \in L^p(\mathbf{R}^3) \bigcap L^2(\mathbf{R}^3)$ with $1 \leq p < \frac{1}{3-2\alpha}$. Can we establish the decay estimate for this case?

In the following, we consider the time decay rate of solutions to systems (1). The motivation is to understand how the parameter α affects the time decay rate of its solutions. Here, we contrast with the generalized heat equations, study the decay rate of solutions for systems (1), establish the L^2 decay of solutions for $u_0 \in L^2(\mathbf{R}^3) \bigcap L^1(\mathbf{R}^3)$ and $0 < \alpha < 2$. More precisely, we have the following result:

Theorem 4. *Suppose that* $0 < \alpha < 2$, $u_0 \in L^2(\mathbf{R}^3) \bigcap L^1(\mathbf{R}^3)$ *and* $\nabla \cdot u_0 = 0$. *Then, for the solution* $u(x, t)$ *of system (1), there exists a positive constant* $C = C(\alpha, \|u_0\|_{L^1}, \|u_0\|_{L^2})$, *such that*

$$\|u(x, t)\|^2_{L^2} \leq C(1+t)^{-\frac{3}{2\alpha}}, \quad \textit{for large } t.$$

Remark 5. The time decay problem of solutions to the dissipative equations implies that the trivial solution is asymptotically stable. It is an interesting problem to consider the time decay rate of solutions to dissipative equations. One of the powerful tools is Fourier splitting method, which is introduced by Schonbek in 1980s (see [24,25]). There are also some other tools to study the decay rate, for example, Kato's method [5,26,27], the maximal principle [28], Zhou's method [29,30] and so on.

Remark 6. As we know, Theorem 4 is the first decay estimates for system (1) with $\alpha \in [\frac{5}{4}, 2)$. However, because of our decay estimate is dependent on $\|\nabla v\|_{L^\infty} \leq Ct^{\frac{2}{\alpha}}$ heavily, it is difficult to break the restriction $\alpha \geq 2$.

The rest of the paper is organized as follows. In the next section, we prove Theorem 1 while the proof of Theorem 4 is postponed in Section 3.

2. Proof of Theorem 1

First of all, we write down the definition of weak solutions for system (1).