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# An image denoising model based on a fourth-order nonlinear partial differential equation

Abdelgader Siddig [a,b], Zhichang Guo [a,*], Zhenyu Zhou [a], Boying Wu [a]

[a] Department of Mathematics, Harbin Institute of Technology, Harbin, 150001, China
[b] Department of Mathematics, University of Albutana, Ruffa'a, 420, Sudan

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ABSTRACT

Image denoising is a challenging task in the fields of image processing and computer vision. Inspired by the good performance of nonlinear fourth-order models in recovering smooth region, we proposed a fourth-order image denoising model. Using the fixed point theorem, we established the existence and uniqueness of the entropy solution. Based on the fast explicit diffusion scheme (FED), numerical experiments illustrate the effectiveness of the suggested method in image denoising. The results have been compared with three famous fourth-order models, You and Kaveh (YK) model, Lysaker, Lundervold and Tai (LLT) model and the more recent mean curvature (MC) model. The proposed model has the superiority in terms of removing noise while preserving image features.



## 1. Introduction

For the past decades, the use of partial differential equations (PDEs) in image restoration has become an effective tool. The main idea is to represent an image as a function $u$ in $\mathbb{R}^2$. This function usually satisfies a time dependent PDE that characterizes the given problem. The solution of this PDE produces the processed image at the scale $t$.

Several nonlinear second-order diffusion-based techniques, in the PDEs or variational forms, have been employed in image denoising since Perona and Malik (PM) introduced their anisotropic diffusion scheme [1] in 1990. The model can be written as

$$\frac{\partial u}{\partial t} - \nabla(g(|\nabla u|^2)\nabla u) = 0, \tag{1.1}$$

where $g(s^2)$ is chosen such that $g(s^2) \to 0$ when $s \to \infty$, and $g(s^2) \to 1$ when $s \to 0$. The function $g$ is the diffusivity function and $|\nabla u|$ is the edge detector. In [2,3], the authors proposed adaptive versions of PM equation. Actually, in [2], they selected the function $g$ as convolution of a Gaussian kernel $G_\sigma$ and the image gradient $\nabla u$. So, the model is based on the equation

$$\frac{\partial u}{\partial t} - \nabla(g(|G_\sigma * \nabla u|^2)\nabla u) = 0. \tag{1.2}$$

And in [3], the authors offered two adaptive version of PM model, the first model is $\alpha$-PM equation, which has the following form

$$\frac{\partial u}{\partial t} = \nabla\left(\frac{\nabla u}{1+(|\nabla u|/K)^\alpha}\right), \tag{1.3}$$

* Corresponding author.
*E-mail address:* gzcwdg@163.com (Z. Guo).

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where $0 \leq \alpha \leq 2$ is a fixed constant. The second one is $\alpha(x)$-PM

$$\frac{\partial u}{\partial t} = \nabla\left(\frac{\nabla u}{1+(|\nabla u|/K)^{\alpha(x)}}\right), \tag{1.4}$$

where $\alpha(x)$ is chosen as

$$\alpha(|\nabla G_\sigma * u_0|) = 2 - \frac{2}{1+k|\nabla G_\sigma * u_0|^2}, \tag{1.5}$$

or

$$\alpha(|\nabla G_\sigma * u|) = 2 - \frac{2}{1+k|\nabla G_\sigma * u|^2}. \tag{1.6}$$

The total variation (TV) norm in [4] is a regularization functional of the form

$$\mathrm{TV} = \int_\Omega |\nabla u| dx. \tag{1.7}$$

Various modifications have been advised to overcome the drawback of the TV model. Strong and Chan in [5] put forward an adaptive TV based regularization model of the form

$$I(u) = \int_\Omega \alpha(x)|\nabla u(x)| dx, \tag{1.8}$$

where the factor $\alpha(x)$ is designed to control the diffusion speed. Another model was introduced by Blomgren et al. [6], where they suggested a denoising functional of the form

$$I(u) = \int_\Omega |\nabla u|^{p(|\nabla u|)} dx, \tag{1.9}$$

where $p(s)$ is a nondecreasing function, $p(s) \to 1$ when $s \to \infty$, and $p(s) \to 2$ when $s \to 0$. For other modification models, we refer the reader to [7,8].

The differences between these second-order models lie in the diffusion speed, which described by controlling coefficients. These coefficients are defined as functions of the gradient magnitude [9]. These second-order PDEs denoising models may cause staircase effects. However, they successfully overcome the image blurring and preserve the features by performing the filtering along but not across the boundaries.

To avoid problems triggered by second-order models such as the staircase phenomenon, You and Kaveh [10] have used the Laplacian of the image, instead of the gradient, to establish a fourth-order model

$$u_t = -\Delta(g(\Delta u)\Delta u), \tag{1.10}$$

where $\Delta$ denotes the Laplacian operator and $g(s) = k^2/(k^2 + s^2)$ and $k$ is an image dependent parameter. Since then, fourth-order PDEs have been widely used in image denoising [11–21]. The fourth-order PDEs have a superiority over the second-order in terms of recovering smooth regions.

In this work, we introduced a fourth-order model to address the problem of denoising images contaminated with additive Gaussian noise. The motivation for proposing this model is to overcome the impediments of second-order models in recovering smooth regions and better preserve the image details. The model solves a nonlinear fourth-order degenerate equation with the noisy image as its initial data. We applied the fixed point method to prove the existence of the solution. Although this technique is common in other areas of applied mathematics, it is not easily found in the image processing literature. Furthermore, in the numerical part, we utilized a novel fast scheme to accelerate the whole numerical implementation of our fourth-order model. This so-called Fast Explicit Diffusion (FED) scheme is proposed by Greweing et al. [22], which is stable, fast and an easy method to implement. To the best of our knowledge, this is the first fourth-order denoising model that implemented using FED scheme. The results obtained by our model demonstrate that the proposed model ranks highest among other three fourth-order models.

The rest of this article is organized as follows. In Section 2, we state some preliminaries that we will use in what follows. Section 3 is devoted to the proposed model and the proofs of existence and uniqueness of its solution. The numerical schemes are presented in Section 4. Numerical results are presented in Section 5, where we have applied our algorithm to two test images and demonstrated the effectiveness of our model in image denoising and the paper is concluded in Section 6.

## 2. Preliminaries

In this section, we recall some necessary definitions and notations [12,23,24]. We begin with some definitions of the space $BV^2$, which consists of functions $u \in W^{1,1}(\Omega)$ s.t $\nabla u \in BV(\Omega)$. This space is also denoted by $BH(\Omega)$. To know more about space of bounded Hessian, we refer the reader to [25–29,42].

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