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A simple empirical formula of origin intensity factor in singular boundary method for two-dimensional Hausdorff derivative Laplace equations with Dirichlet boundary

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ABSTRACT

This paper presents a simple empirical formula of origin intensity factor in singular boundary method (SBM) solution of Hausdorff derivative Laplace equations. The SBM with the empirical formula is mathematically more simple and computationally more efficient than using the other techniques for origin intensity factor. Numerical experiments simulate the steady heat conduction through fractal media governed by the Hausdorff Laplace equation, and show the efficiency and reliability benefits of the present SBM empirical formula.

1. Introduction

The Hausdorff derivative [1], as a local operator, is introduced to overcome high computing costs of the non-local fractional derivative, and in recent years has widely been applied to various complex problems, such as water transport in unsaturated media [2], heat transfer of Li-ion cells [3], magnetic resonance imaging [4,5], and economics [6]. The Hausdorff derivative can be used to describe the anomalous diffusion problems underlying the well-known stretched Gaussian statistics and the Kohlrausch–Williams–Watts stretched exponential decay. In addition, its derivative order has clear physical meaning and is directly related to the Hausdorff fractal dimension [7,8]. In fact, although the Hausdorff and fractal derivatives are both metric derivatives (as pointed in Ref. [9]), their definitions are based on quite different metrics. Consequently, the properties of these derivatives are significantly different [10,11]. Specifically, there is no way to formulate non-trivial boundary conditions with the use of the fractal derivative introduced in Ref. [8], because, per its definition, the fractal derivative is singular at the fractal boundary [10]. The rigorous definition of the Laplace operator associated with the Hausdorff derivative is given in Ref. [11]. More general forms of the Laplace operator associated with fractal metrics are discussed in Ref. [12]. In addition, the conformable derivative [13] can simply be transformed to the Hausdorff derivative and their equivalence has been numerically verified [14,15].

The Hausdorff derivative underlies a non-Euclidean metric, called the Hausdorff fractal distance [16]. It should be noted that the Euclidean distance is only a special limiting case of the Hausdorff fractal distance. Chen and Wang [16] gave the fundamental solutions of a few typical Hausdorff differential operators via the Hausdorff fractal distance, and employed the singular boundary method (SBM) [17–19], a recent meshless boundary collocation method based on the fundamental

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solutions, to simulate the potential problems governed by the Hausdorff Laplace equations. The SBM takes the fundamental solution as the basis functions and introduces the concept of the origin intensity factor (OIF) to remove the singularities of fundamental solution upon the coincidence of source and collocation nodes on the physical boundary. As shown in Refs. [20,21], the OIF plays a central role in guaranteeing numerical accuracy and efficiency, and therefore considerable studies have been made to design efficient techniques to determine the OIF, such as the inverse interpolation technique, the subtracting and adding-back desingularization technique, the integral average approach, and the empirical formulas. For details see Ref. [21] and references therein. Due to the fractal characteristics of the Hausdorff derivative partial differential equations, it is not an easy task to evaluate the OIF in the SBM for such equations.

For the integer-order dimensional potential problems, Refs. [20] and [21] present the efficient empirical formulas of two- and three-dimensional cases, respectively. As far as we know, the empirical formula is undoubtedly the simplest and most efficient technique to make the SBM simulation. In this regard, this paper proposes an empirical formula of the OIF in the SBM for two-dimensional Hausdorff derivative Laplace equations, with the help of the scale transformation and the Hausdorff fractal distance. Compared with the other techniques, the empirical formula can be easily used in the Hausdorff derivative Laplace equation. Numerical experiments confirm the efficiency and accuracy of the proposed empirical formula.

The rest of the paper is organized as follows. The Hausdorff Laplace equation and its fundamental solution are introduced in Section 2. The empirical formula of the OIF in the SBM for Hausdorff Laplace equations is presented in Section 3. In Section 4, we discuss numerical experiment results of steady heat conduction governed by the Hausdorff Laplace equation via the SBM. Some conclusions are made in Section 5.

2. Hausdorff Laplacian and its fundamental solution

2.1. Hausdorff Laplacian

Considering a particle movement in the fractal time direction, the movement displacement can be expressed as [16]

$$\ell(\tau) = v(\tau - t_0)^{\alpha}, \tag{1}$$

where ℓ denotes the distance, v represents the velocity, τ the current time instance, t_0 the initial instance, α the fractal dimension in time. If the velocity v varies with the time, the Hausdorff integral distance can be computed by the following integral formula

$$\ell(t) = \int_{t_0}^{t} v(\tau)\, d(\tau - t_0)^{\alpha}. \tag{2}$$

Based on the above expression, we can get the following Hausdorff derivative:

$$\frac{d\ell}{dt^{\alpha}} = \lim_{\bar{t}\to t} \frac{\ell(t) - \ell(\bar{t})}{(t - t_0)^{\alpha} - (\bar{t} - t_0)^{\alpha}} = \frac{1}{\alpha(t - t_0)^{\alpha-1}} \frac{d\ell}{dt}. \tag{3}$$

Let the initial instance be $t_0 = 0$, Eq. (3) is rewritten as

$$\frac{d\ell}{dt^{\alpha}} = \lim_{\bar{t}\to t} \frac{\ell(t) - \ell(\bar{t})}{t^{\alpha} - \bar{t}^{\alpha}} = \frac{1}{\alpha t^{\alpha-1}} \frac{d\ell}{dt}. \tag{4}$$

Similarly, the Hausdorff derivative in space is stated as

$$\frac{du}{dx^{\beta}} = \lim_{\bar{x}\to x} \frac{u(x) - u(x)}{x^{\beta} - \bar{x}^{\beta}} = \frac{1}{\beta x^{\beta-1}} \frac{du}{dx}, \tag{5}$$

where u denotes the physics quantity in a fractal medium such as temperature and displacement, $0 < \beta \le 1$ represents the Hausdorff fractal in space. Note that the origin of spatial coordinate system in Eq. (5) is assumed zero.

The Hausdorff derivative Laplace equation in two topological dimension is given by

$$\frac{\partial}{\partial x^{\beta}}\left(\frac{\partial u}{\partial x^{\beta}}\right) + \frac{\partial}{\partial y^{\beta}}\left(\frac{\partial u}{\partial y^{\beta}}\right) = 0. \tag{6}$$

2.2. Hausdorff fundamental solutions

In order to develop the partial differential equation model of physical problems on nowhere integer-order differentiable fractal, we need to map the fractal problem into a continuum framework with a proper fractal metric. The Hausdorff fractal distance, as a non-Euclidean metric, is presented to describe fractal media in Refs. [1,7,16]:

$$\begin{cases} \Delta t^{\alpha} = t^{\alpha} - t_0^{\alpha} \\ r^{\beta} = \sqrt{\sum_{i=1}^{d} \left(x_i^{\beta} - y_i^{\beta}\right)^2}, \end{cases} \tag{7}$$

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