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Spectral approximation for polynomial eigenvalue problems

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ABSTRACT

In this paper, we extent the classical spectral approximation theory for compact and bounded operators to general linear operators, and then apply it to polynomial eigenvalue problems (PEP). We also study the essential spectrum in PEPs, and prove that this spectrum is stable under relatively compact perturbations. Based on this analysis, we give some suggestions to make algorithms for solving PEPs more efficient.

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1. Introduction

Polynomial eigenvalue problems (PEP) arise in many applications, e.g. acoustics, fluid mechanics and photonic crystals. There exists an extensive literature discussing PEPs and related numerical methods [1–3]. In this paper, we will study the approximation theory for PEPs and the behavior of their essential spectra under relatively compact perturbations. The analysis shows that it is reasonable in eigenvalue computations to exclude a small neighborhood around the points in the essential spectrum. These eigenvalues are densely clustered and hard to compute, while they do not provide much information compared to well separated eigenvalues.

Let X be a Banach space equipped with the norm $\|\cdot\|$, and $\{M_i\}_{i=0}^{p-1}$ be a series of linear operators from X to itself. We consider a polynomial eigenvalue problem (PEP): find $(\lambda, u) \in \mathbb{C} \times X$, with $u \not\equiv 0$, such that

$$\mathcal{P}(\lambda)u \equiv \left(\lambda^p + \sum_{i=0}^{p-1} \lambda^i M_i\right) u = 0. \tag{1.1}$$

In applications, many problems can be translated into this form. Let $\mathbb{X} = \underbrace{X \times \cdots \times X}_{p}$ be equipped with the norm $\|\boldsymbol{u}\| = \|u_0\| + \cdots + \|u_{p-1}\|$ for $\boldsymbol{u} = (u_0, \ldots, u_{p-1})^T \in \mathbb{X}$. We consider the linear operator $\mathcal{S} : \mathbb{X} \to \mathbb{X}$ defined as

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$$\mathcal{S} = \begin{pmatrix} 0 & 0 & \cdots & \cdots & -M_0 \\ I & 0 & \cdots & \cdots & -M_1 \\ 0 & I & \ddots & \cdots & -M_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & -M_{p-1} \end{pmatrix},$$

with I the identity operator on X. If (λ, u) is an eigenpair of (1.1), then $(\lambda, \boldsymbol{u})$ is a pair satisfying

$$\mathcal{S}\boldsymbol{u} = \lambda \boldsymbol{u}, \tag{1.2}$$

which follows directly from the relations

$$\begin{aligned} &u_{p-1} = u, \\ &u_{p-i} = \lambda u_{p-i+1} + M_{p-i+1} u_{p-1}, \quad \text{for } i = 2, \ldots, p, \end{aligned} \tag{1.3}$$

obtained by expanding (1.2). This means that (1.2) and (1.1) share the same spectrum [4]. The eigenvalue problem (1.2) is called the linearized formulation of (1.1). We consider the spectrum of (1.2) and its approximation instead of (1.1). The resolvent set of $\mathcal{S}$ is denoted by $\rho(\mathcal{S})$, and its spectrum is $\sigma(\mathcal{S}) = \mathbb{C} \setminus \rho(\mathcal{S})$.

Classical spectral approximation theories are developed for compact operators [5] and bounded operators [6]. The crucial point in these theories is uniform (or norm) convergence of approximate operators. For example, when studying the spectral approximation of the Laplace operator numerically, what we study, in fact, is its inverse (or solution) operator, which is a compact operator. In the eigenvalue problem (1.2), it is, however, possible that some of the operators $\{M_i\}_{i=0}^{p-1}$ are unbounded, which results in $\mathcal{S}$ being an unbounded operator. The inverse of $\mathcal{S}$ is generally not easy to obtain, and it may be also unbounded. This motivates us to analyze more general operators, and use this theory to approximate their spectrum.

In many (polynomial) eigenvalue problems, a large number of eigenvalues will be concentrated in some small regions around points in the essential spectrum in the complex plane $\mathbb{C}$. This can complicate the numerical solution of these eigenvalue problems. We present two examples to illustrate this phenomenon. The first example is: find $(\lambda, u) \in \mathbb{C} \times X$ with $u \not\equiv 0$, such that

$$\lambda^2 Iu - 3\lambda Iu + 2Iu = 0. \tag{1.4}$$

This problem only contains two eigenvalues, $\lambda_1 = 1$ and $\lambda_2 = 2$, each with multiplicity equal to $\dim X$. For the second example, we consider the quadratic eigenvalue problem

$$\left(\lambda^2 - 2\lambda A + A\right) u = 0. \tag{1.5}$$

Here, $A : L^2(\Omega) \to L^2(\Omega)$, with $\Omega = [0, \pi]$, is the one-dimensional Laplace operator, such that for $f \in L^2(\Omega)$, $Au = f \in L^2(\Omega)$ satisfies

$$\begin{aligned} -u''(x) &= f \qquad \text{in} \quad \Omega, \\ u(0) = u(\pi) &= 0. \end{aligned} \tag{1.6}$$

The set of exact eigenpairs of A are $\{(k^2, \sin kx)\}_{k=1}^{+\infty}$. The quadratic eigenvalue problem (1.5) has the same set of eigenfunctions as A. Introducing the pair $(k^2, \sin kx)$ into (1.5), we have

$$\lambda_k^2 - 2k^2\lambda_k + k^2 = 0. \tag{1.7}$$

The eigenvalues of (1.5) then are equal to

$$\lambda^{(k,1)} = k^2 - \sqrt{k^4 - k^2} \quad \text{and} \quad \lambda^{(k,2)} = k^2 + \sqrt{k^4 - k^2} \quad \text{for } k = 1, 2, \ldots. \tag{1.8}$$

For $k \to +\infty$, $\lambda^{(k,2)}$ are isolated points tending to $+\infty$, while $\lambda^{(k,1)} \to \frac{1}{2}$. The eigenvalues $\lambda = 1, 2$ (infinite multiplicity) in (1.4), and $\lambda = \frac{1}{2}, +\infty$ (accumulation points) in (1.5) are essential spectral points. If we discretize I in (1.4) using the identity matrix $I_{n\times n}$ the multiplicities of the two numerical eigenvalues $\lambda = 1, 2$ are both n. For the discrete approximation of (1.5), there will be a large number of numerical eigenvalues around $\frac{1}{2}$ in the complex plane, as shown in Fig. 1.1. In computations, we should avoid computing too many numerical eigenvalues close to these essential spectral points, which are expensive to compute and provide little information compared with well isolated eigenvalues.

For arbitrary PEPs, it is, however, not easy to compute the essential spectral points analytically, as could be done for Problems (1.4) and (1.5). In particular, if the operators $\{M_i\}_{i=0}^{p-1}$ are different from each other. For finite-dimensional operators, there are no accumulation points or points with infinite multiplicity in their spectrum. So we cannot obtain the essential spectra from their corresponding discrete problems, either.

In some applications [7,2,8], each M_i in the PEP may be split into two parts, one 'main' operator A and perturbations B_i, i.e.

$$M_i = \alpha_i A + B_i \quad \text{for} \quad i = 0, \ldots, p-1, \quad \text{where} \quad \alpha_i \in \mathbb{C}.$$

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