ARTICLE IN PRESS

Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

On the numerical solution of high order multi-dimensional elliptic PDEs

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ARTICLE INFO

Article history:
Received 21 January 2018
Received in revised form 22 May 2018
Accepted 9 June 2018
Available online xxxx

Keywords:
Differential quadrature method
Elliptic PDEs
Biharmonic equation
Triharmonic equation
Error bounds

ABSTRACT

Using the idea of differential quadrature, two new methods are constructed to approximate the solution of elliptic partial differential equations in higher dimensions. To obtain the weighting coefficients of the first method, a mixture of grid points and mid points of the uniform partition is used. The order of convergence obtained by the first algorithm is non-optimal, so we mixed the idea with spline collocation to obtain higher order approximations. Using the weighting coefficients of the non-optimal algorithm, some new weighting coefficients are obtained which are used to obtain higher accuracy. For the first time, a sixth order approximation is obtained to the solution of well-known high order multi-dimensional elliptic PDEs such as biharmonic and triharmonic problems. We found a way to increase the order of convergence and the accuracy without increasing the CPU runtime. The block form of the coefficients matrix for the linear case is presented to have a better vision on the system arised from the method. Some examples of biharmonic and triharmonic problems are solved to show the good performance and applicability of the proposed algorithms.

1. Introduction

Consider the following elliptic problem

$$\nabla^{2k}\psi(x) = f(x, \psi(x), \nabla^2\psi(x), \ldots, \nabla^{2k-2}\psi(x)), \qquad x \in \Omega, \tag{1.1}$$

along with appropriate Dirichlet or Neumann boundary conditions

$$B\psi(x) = g, \qquad x \in \partial\Omega, \tag{1.2}$$

where $k \geq 1$, $\Omega \subset \mathbb{R}^d$, B is the boundary differential operator defined on $\partial\Omega$, and ∇^2 is the $d-$dimensional Laplace operator

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_d^2}.$$

Suppose that the problem possess a unique solution and the partial derivatives of the solution are existed and commutative. Also suppose that the separation of variables is valid for the problem. The cases with $k = 1, 2, 3$, are of more interest because of their many applications in science and engineering. For $k = 1$, the problem is reduced to the well-known Poisson equation which can be written in the following form

$$\nabla^2\psi(x) = f(x, \psi(x)), \qquad x \in \Omega. \tag{1.3}$$

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https://doi.org/10.1016/j.camwa.2018.06.017
0898-1221/

Please cite this article in press as: M. Ghasemi, On the numerical solution of high order multi-dimensional elliptic PDEs, Computers and Mathematics with Applications (2018), https://doi.org/10.1016/j.camwa.2018.06.017.

This problem is also known as harmonic problem. Many kinds of physical phenomena including problems related to the potential fields, fluid dynamics [1], acoustics [2] and electromagnetic [3] can be described by Poisson equation. For $k = 2$, problem (1.1) represents the biharmonic equation which arises in various areas in mechanics such as theory of elasticity and the deformation of elastic plates [4]. The two-dimensional biharmonic equation also can be used in modeling some problems such as bending of thin elastic clamped rectangular plates, equilibrium of an elastic rectangle and flow of very viscous fluid in rectangular cavity [5]. For $k = 3$, problem (1.1) reduces to a sixth order elliptic PDE which is well-known as triharmonic equation. The triharmonic equation arises in the modeling of ulcers [6], viscous fluid [7], computer graphics [8], and interactive design [9].

Many authors have tried to approximate the solution of problem (1.1) for various values of k. Several methods such as cyclic reduction method [10], domain decomposition [11] and multigrid techniques [12] have been developed to the solution of Poisson equation. Gupta et al. have introduced two different algorithms for the solution of 2D Poisson equation [13]. They compared the results obtained from the second and fourth order methods together to show the efficiency of their fourth order algorithm. In [14], a fourth order finite difference with multigrid algorithm was developed to solve the three dimensional Poisson equation. In the past few decades, many different approaches have been introduced in the literatures to the solution of biharmonic problem. Some approaches were based on transforming the original problem into a system of coupled Poisson equations which then can be solved by any proper discretization technique. Such a technique have been used in [15] based on finite differences and in [16,17] based on spectral elements and Galerkin discretization methodologies. To the best of author's knowledge, there exist only few papers on studding numerical techniques for triharmonic equation. The boundary operator method along with parametric extrapolation technique has been used to the solution of triharmonic equation by Quan [18]. In [19,20], an iterative method has been constructed to approximate the solution of triharmonic equation. The triharmonic equation has been studied by Mohanty et al. using various kinds of numerical methods including finite differences of various orders and single cell Numerov type discretization techniques [21–25].

In [26], Bellman and Casti have introduced a powerful algorithm based on Gauss quadrature idea to approximate the solution of partial differential equations which is well-known as differential quadrature method (DQM). For any arbitrary partition of the domain, the algorithm uses a weighted summation of all the function values at the whole partition to approximate the derivatives of the function at every point of the partition. So it is needed to choose some grid points as well as some basis functions to be able to obtain the weighting coefficients of the algorithm. Belman et al. used shifted Legendre polynomials and cardinal spline as test functions to find the weighting coefficients [26,27]. Quan and Chang found some explicit formulas for writing coefficients based on Lagrange basis functions [28,29]. Tomasiello [30], provided a comprehensive review on DQ methods with applications to engineering and physical sciences. Many other researchers have been worked on differential quadrature method based on various kinds of basis functions [31–43]. Zhong [34], used the idea of differential quadrature based on cardinal spline. Korkmaz et al. [36–38], Mittal et al. [39–41] and many other researchers have used B-spline basis functions to obtain the weighting coefficients of DQ method. In [44,45] the differential quadrature method based spline quasi-interpolation has been constructed to the solution differential equations and the optimal orders of convergence have been obtained. Spline based DQM has been used to the solution of fractional differential equations in [46]. More recently, a new spline-based differential quadrature algorithm was developed to the solution of multi-dimensional time dependent PDEs [47]. By combining the idea with spline collocation and obtaining the error bounds, the author found a way to obtain optimal convergence using cubic B-spline. In [48], the arbitrary degree spline-based DQM has been used to the solution of partial differential equations of arbitrary order. The orders of convergence was non-optimal.

In the current work, we aim to use B-spline basis functions to obtain the weighting coefficients of differential quadrature method. Since the number of grid points are not equal with the number of basis functions, we construct our method on a set of data points combined from grid points and mid points of a uniform partition. At first the weighting coefficients are obtained, then by introducing some discrete operators and obtaining the new weighting coefficients, higher order approximations are constructed. The important point is that, we improve the accuracy from without increasing the CPU runtime. The method is used to approximate the solution of multi-dimensional elliptic PDEs including biharmonic and triharmonic equations. Also the matrix representation of the method for linear cases is included in the paper to have a better vision on the system raised from DQM.

The organization of paper is as follows: in Section 2 after introducing the idea of differential quadrature, a new approximation based on B-spline functions will be introduced. Then the new method will be used to approximate the derivatives of functions with one and two variables and the error bounds will be obtained. Section 3 is devoted to obtaining higher orders of accuracy using the proposed method. By introducing some new weighting coefficients, the new method will be constructed which is more accurate than the first one but its runtime is the same. In Section 3 the proposed methods will be used to approximate the solution of multi-dimensional elliptic PDEs of different orders. Matrix representation of the system arised from the method for linear elliptic problems will be introduced in Section 4. Finally in Section 5 we will solve some various problems of biharmonic and triharmonic equation. The results will be compared with other existing methods to show the applicability and efficiency of our proposed algorithms.

2. Differential quadrature method

2.1. Non-optimal approximations

In order to describe the idea of differential quadrature method we consider the sufficiently smooth function $\psi(x)$ defined on $[a, b]$, and a uniform partition on the function's domain with the grid points x_i, $i = 0(1)n$. The rth order derivative of $\psi(x)$