Physics Letters A ••• (••••) •••–•••

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

The equivalence of Bell's inequality and the Nash inequality in a quantum game-theoretic setting

Azhar Iqbal, James M. Chappell, Derek Abbott

School of Electrical & Electronic Engineering, University of Adelaide, South Australia 5005, Australia

A R T I C L E   I N F O

Article history:
Received 15 January 2018
Received in revised form 5 July 2018
Accepted 14 August 2018
Available online xxxx
Communicated by M.G.A. Paris

Keywords:
Quantum games
EPR experiment
Nash equilibrium
Bell's inequality

A B S T R A C T

The interaction of competing agents is described by classical game theory. It is now well known that this can be extended to the quantum domain, where agents obey the rules of quantum mechanics. This is of emerging interest for exploring quantum foundations, quantum protocols, quantum auctions, quantum cryptography, and the dynamics of quantum cryptocurrency, for example. In this paper, we investigate two-player games in which a strategy pair can exist as a Nash equilibrium when the games obey the rules of quantum mechanics. Using a generalized Einstein–Podolsky–Rosen (EPR) setting for two-player quantum games, and considering a particular strategy pair, we identify sets of games for which the pair can exist as a Nash equilibrium only when Bell's inequality is violated. We thus determine specific games for which the Nash inequality becomes equivalent to Bell's inequality for the considered strategy pair.

1. Introduction

A quantum game [1–5] describes the strategic interaction among a set of players sharing quantum states. Players' strategic choices, or strategies [6–8], are local unitary transformations on the quantum state. The state evolves unitarily and finally the players' payoffs, or utilities, are obtained by measuring the entangled state. It turns out that under certain situations sharing of an entangled quantum state can put the players in an advantageous position and more efficient outcomes of the game can then emerge. For readers not familiar with the formalism of quantum theory [9], sharing an entangled state can be considered equivalent to the situation in which the players have (shared) access to a 'quantum system' having some intrinsically non-classical aspects. A quantum game would then involve a strategic manoeuvring of the shared quantum system in which different and perhaps more efficient outcome(s) of the game can emerge due to non-classical aspect(s) of the shared system.

Now, it is well known that non-classical, and thus apparently strange, aspects of a shared quantum system can be expressed as constraints on probabilities relevant to the shared system. Usually expressed as constraints in correlations, the famous Bell's inequality [9–14] can be re-expressed as constraints on the relevant joint probability and its marginals [15–18]. Essentially, Bell's inequality emerges as being the necessary and sufficient condition requiring a joint probability distribution to exist given a set of marginals. It is well known that Bell's inequality can be violated by a set of quantum mechanical probabilities—the probabilities that are obtained by the quantum probability rule. This turns out to be the case even though the quantum probabilities are normalized as the classical probabilities are. This is because for a set of marginal (quantum) probabilities that are obtained via the quantum probability rule, the corresponding joint probability distribution may not exist. The possibility to express truly non-classical aspects of a quantum system in only probabilistic terms [19] has led to suggestions for schemes of quantum games [26–28,30,33–35] that do not refer to quantum states, unitary transformations, and/or the quantum measurement.

In a classical game allowing mixed strategies, the players' strategies are convex linear combinations, with real coefficients, of their pure strategies [8]. Players' strategies in a quantum game [2,3], however, are unitary transformations and thus belong to much larger strategy spaces. This led to the arguments [36] that quantum games can perhaps be viewed as extended classical games. In order to obtain an improved comparison between classical and quantum games, it was suggested [24] that the players' strategy sets need to be identical. This has motivated proposals [33–35] of quantum games in which players' strategies are classical, as being convex linear combinations (with real coefficients) of the classical strategies, and the quantum game emerges from the non-classical aspects of a shared probabilistic physical system—as expressed by the constraints on relevant probabilities and their marginals [15–18].

E-mail address: mr.azhariqbal@gmail.com (A. Iqbal).

https://doi.org/10.1016/j.physleta.2018.08.011

In the usual approach in the area of quantum games [5], a classical game is defined, or given, at the start and its quantum version is developed afterwards. The usual reasonable requirement being that the classical mixed-strategy game can be recovered from the quantum game. One then studies whether the quantum game offers any non-classical outcome(s). In this paper, the players' strategies in the quantum game remain classical whereas the new quantum, or non-classical, outcome(s) of the game emerge from the peculiar quantum probabilities relevant to the quantum system that two players share to play the game. In contrast to the usual approach in quantum games, in which the players' strategies are unitary transformations, here we consider a particular classical strategy pair and then enquire about the set of games for which that strategy pair can exist as a Nash equilibrium (NE) [6–8]. In particular, for a given strategy pair, we investigate whether there are such games for which that strategy pair can exist as a NE only when the corresponding Bell's inequality is violated by the quantum probabilities relevant to the shared quantum system.

We consider two-player games that can be played using the setting of generalized Einstein–Podolsky–Rosen (EPR) experiments [9,14,19]. As is known, in this setting a probabilistic version of Bell's inequality can be obtained [15–19]. We consider particular strategies and find the sets of games for which the strategies can exist as a NE only when Bell's inequality is violated. By identifying such games, we show that there exist strategic outcomes that can only be realized when the game is played quantum mechanically and also only when the corresponding Bell's inequality is violated.

The connection between Bell's inequality and the NE was originally reported in Ref. [20]. However, the Ref. [20] did not use an EPR setting. In the present paper, we show that the mentioned connection becomes explicitly direct by using an EPR setting in playing a quantum game.

2. Two-player quantum games using the EPR experiment setting

The EPR setting for playing quantum games was introduced in Ref. [24] and was further investigated in Refs. [25–32]. The Refs. [26–28,30,33–35] investigate using the setting of generalized EPR experiments [19] for playing quantum games. This setting permits consideration of a probabilistic version of the corresponding Bell's inequality, which allows construction of quantum games without referring to the mathematical formalism of quantum mechanics including Hilbert space, unitary transformations, entangling operations, and quantum measurements [9]. The relationship between the NE and aspects of Bell's inequality have been indicated in Refs. [21–23]. The present paper's contribution consists of bringing into focus this relationship and, in particular, finding the specific games for which this relationship can be explicitly defined. Moreover, in order to achieve this the present paper uses EPR setting and the probabilistic version of Bell's inequality.

In the setting of the generalized EPR experiment, Alice and Bob are spatially separated and are unable to communicate with each other. In an individual run, both receive one half of a pair of particles originating from a common source. In the same run of the experiment, both players choose one from two given (pure) strategies. These strategies are the two directions in space along which spin or polarization measurements can be made. We denote these directions to be S_1, S_2 for Alice and S_1', S_2' for Bob. Each measurement generates $+1$ or -1 as the outcome. Experimental results are recorded for a large number of individual runs of the experiment. Payoffs are then awarded that depend on the directions the players choose over many runs (defining the players' strategies), the matrix of the game they play, and the statistics of the measurement outcomes. For instance, we denote $\Pr(+1,+1;S_1,S_1')$ as the probability of both Alice and Bob obtaining $+1$ when Alice selects the direction S_1 whereas Bob selects the direction S_1'. We write ϵ_1 for the probability $\Pr(+1,+1;S_1,S_1')$ and ϵ_8 for the probability $\Pr(-1,-1;S_1,S_2')$ and likewise one can then write down the relevant probabilities as

$$\text{Alice}\;\begin{array}{cc} \\ S_1 & \begin{array}{c}+1\\-1\end{array} \\ S_2 & \begin{array}{c}+1\\-1\end{array}\end{array}\overset{\text{Bob}}{\begin{array}{cc|cc} \multicolumn{2}{c}{S_1'} & \multicolumn{2}{c}{S_2'} \\ +1 & -1 & +1 & -1 \\ \epsilon_1 & \epsilon_2 & \epsilon_5 & \epsilon_6 \\ \epsilon_3 & \epsilon_4 & \epsilon_7 & \epsilon_8 \\ \hline \epsilon_9 & \epsilon_{10} & \epsilon_{13} & \epsilon_{14} \\ \epsilon_{11} & \epsilon_{12} & \epsilon_{15} & \epsilon_{16} \end{array}}. \tag{1}$$

Being normalized, EPR probabilities ϵ_i satisfy the relations

$$\Sigma_{i=1}^{4}\epsilon_i = 1,\ \Sigma_{i=5}^{8}\epsilon_i = 1,\ \Sigma_{i=9}^{12}\epsilon_i = 1,\ \Sigma_{i=13}^{16}\epsilon_i = 1. \tag{2}$$

Consider in (1), for instance, the case when Alice plays her strategy S_2 and Bob plays his strategy S_1'. The two arms of the Stern–Gerlach apparatus are rotated along these two directions and the quantum measurement is performed. According to the above table, the probability that both experimental outcomes are -1 is then ϵ_{12}. Similarly, the probability that the observer 1's experimental outcome is $+1$ and observer 2's experimental outcome is -1 is given by ϵ_{10}. The other entries in (1) can similarly be explained. In the present paper, the EPR setting is enforced and that the players can only choose between two directions.

We now consider a game between two players Alice and Bob, which is defined by the real numbers a_i and b_i for $1 \le i \le 16$, and is given by

$$\text{Alice}\;\begin{array}{c} \\ S_1 \\ \\ S_2 \end{array}\overset{\text{Bob}}{\begin{array}{cc|cc} S_1' & & & S_2' \\ (a_1,b_1) & (a_2,b_2) & (a_5,b_5) & (a_6,b_6) \\ (a_3,b_3) & (a_4,b_4) & (a_7,b_7) & (a_8,b_8) \\ \hline (a_9,b_9) & (a_{10},b_{10}) & (a_{13},b_{13}) & (a_{14},b_{14}) \\ (a_{11},b_{11}) & (a_{12},b_{12}) & (a_{15},b_{15}) & (a_{16},b_{16}) \end{array}}. \tag{3}$$

For this game, we now define the players' pure strategy payoff relations as

$$\Pi_{A,B}(S_1,S_1') = \Sigma_{i=1}^{4}(a_i,b_i)\epsilon_i,\ \Pi_{A,B}(S_1,S_2') = \Sigma_{i=5}^{8}(a_i,b_i)\epsilon_i,$$
$$\Pi_{A,B}(S_2,S_1') = \Sigma_{i=9}^{12}(a_i,b_i)\epsilon_i,\ \Pi_{A,B}(S_2,S_2') = \Sigma_{i=13}^{16}(a_i,b_i)\epsilon_i, \tag{4}$$

where $\Pi_A(S_1,S_2')$, for example, is Alice's payoff when she plays S_1 and Bob plays S_2'.

It can be seen that in the way it is defined, the game is inherently probabilistic. That is, in (3) the players' payoffs even for their pure strategies assume an underlying probability distribution as given by (1). Now, we can also define a mixed-strategy version of this game as follows. Consider Alice playing the strategy S_1 with probability p and the strategy S_2 with probability $(1-p)$ whereas Bob playing the strategy S_1' with probability q and the strategy S_2' with probability $(1-q)$. Using (3), (4) the players' mixed strategy payoff relations can then be obtained as

$$\Pi_{A,B}(p,q) = \begin{pmatrix} p \\ 1-p \end{pmatrix}^T \begin{pmatrix} \Pi_{A,B}(S_1,S_1') & \Pi_{A,B}(S_1,S_2') \\ \Pi_{A,B}(S_2,S_1') & \Pi_{A,B}(S_2,S_2') \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix}. \tag{5}$$

Please cite this article in press as: A. Iqbal et al., The equivalence of Bell's inequality and the Nash inequality in a quantum game-theoretic setting, Phys. Lett. A (2018), https://doi.org/10.1016/j.physleta.2018.08.011