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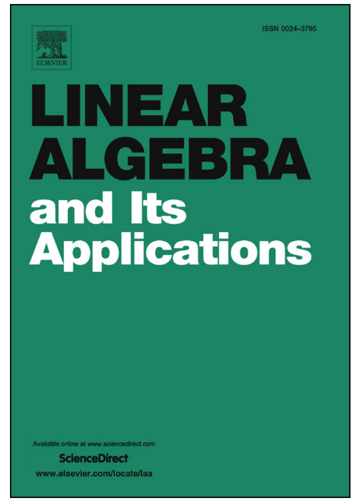

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ESTIMATES OF THE DETERMINANT OF A PERTURBED IDENTITY MATRIX*

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Abstract. Recently Brent et al. presented new estimates for the determinant of a real perturbation $I + E$ of the identity matrix. They give a lower and an upper bound depending on the maximum absolute value of the diagonal and the off-diagonal elements of E, and show that either bound is sharp. Their bounds will always include 1, and the difference of the bounds is at least $\mathrm{tr}(E)$. In this note we present a lower and an upper bound depending on the trace and Frobenius norm $\epsilon := \|E\|_F$ of the (real or complex) perturbation E, where the difference of the bounds is not larger than $\epsilon^2 + \mathcal{O}(\epsilon^3)$ provided that $\epsilon < 1$. Moreover, we prove a bound on the relative error between $\det(I + E)$ and $\exp(\mathrm{tr}(E))$ of order ϵ^2.

Key words. Determinant, Hadamard bound, Ostrowski bound, Hans-Schneider bound, perturbation of identity

AMS subject classifications. 15A15, 65F40

1. Introduction and main results. Classical estimates for the determinant of a matrix include the Hadamard bound [7] or Gershgorin circles [6]. Moreover, Ostrowski [11, 12, 13] gave a number of lower and upper bounds. Other estimates include [4, 9, 1]. In particular, bounds for the determinant of a perturbed identity matrix are given in Ostrowski's papers, or in [15].

Recently, new sharp bounds for $\det(I + E)$ have been presented by Brent et al. in [2, 3]. Denote by δ the maximum absolute value of the diagonal elements, and by ε the maximum absolute value of the off-diagonal elements of a real $n \times n$-matrix E. Then [2, 3] prove

$$(1 - \delta - (n-1)\varepsilon)(1 - \delta + \varepsilon)^{n-1} \le \det(I + E) \le ((1 + \delta)^2 + (n-1)\varepsilon^2)^{n/2}, \tag{1}$$

where $\delta + (n-1)\varepsilon \le 1$ is supposed for the left inequality. Both inequalities are sharp as by explicit examples given in [2, 3]. For convergent E, Fredholm's identity [5]

$$\det(I + E) = \exp\left(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\mathrm{tr}(E^k)}{k}\right) \tag{2}$$

yields $\det(I + E) = \exp(\mathrm{tr}(E)) + \mathcal{O}(\varepsilon^2)$ for $\|E\| \le \varepsilon < 1$ and some matrix norm $\|\cdot\|$. This is reflected in (1). Although being individually sharp, the upper and lower bound in (1) always include the number 1 and differ by at least $\mathrm{tr}(E)$. That is also true for most of the other bounds mentioned.

Notable exceptions are papers by Ostrowski [14] and Hans Schneider [16], proving bounds depending on the trace and on the absolute row sums of E. If all elements of E are bounded by ε in absolute value, then either difference between upper and lower bound is $\mathcal{O}(n^3\varepsilon^2)$.

For real or complex E, we prove two-sided bounds differing by $\mathcal{O}(\epsilon^2)$, where $\epsilon := \|E\|_F = [\mathrm{tr}(E^H E)]^{1/2}$ denotes the Frobenius (or Hilbert-Schmidt) norm. We prove absolute bounds on $|\det(I + E)|$, and relative bounds on $\det(I + E)$.

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