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Relation between the matching number and the second largest distance Laplacian eigenvalue of a graph

Fenglei Tian [a,*,1], Dein Wong [b,1]

[a] School of Management, Institute of Operations Research, Qufu Normal University, Rizhao 276826, China
[b] School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China

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ABSTRACT

Let G be a connected simple graph with matching number $m(G)$. The second largest distance Laplacian eigenvalue of G is denoted by $\partial_2(G)$. In this paper, we investigate the relation between the matching number and the second largest distance Laplacian eigenvalue of G, establishing the lower bounds of $\partial_2(G)$ in terms of $m(G)$. Moreover, all the extremal graphs attaining the lower bounds are completely characterized.

1. Introduction

Let G be a connected simple graph. The distance Laplacian matrix and distance signless Laplacian matrix of G are defined as $\mathcal{L}(G) = diag(Tr) - \mathcal{D}(G)$ and $\mathcal{Q}(G) =$

* Corresponding author.
E-mail address: tflqsd@qfnu.edu.cn (F. Tian).
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$diag(Tr) + \mathcal{D}(G)$ respectively, where $\mathcal{D}(G)$ is the distance matrix of G and $diag(Tr)$ is a diagonal matrix of vertex transmissions of G. Since the two matrices were defined by Aouchiche and Hansen in [1], they have been investigated intensively. In particular, many papers (see [2,3,6,8–15]) deal with the largest eigenvalues of the two matrices, which are called distance Laplacian spectral radius and distance signless Laplacian spectral radius, respectively. As a consequence, many families of graphs with minimum (or maximum) distance Laplacian (or distance signless Laplacian) spectral radius are characterized.

However, compared with the largest eigenvalues of the two matrices, the second largest eigenvalues of $\mathcal{L}(G)$ and $\mathcal{Q}(G)$, called the second largest distance Laplacian and distance signless Laplacian eigenvalues respectively, have not been studied extensively. The followings are some known conclusions. For any n-vertex graph G and tree T, Tian et al. [16] proved that the second largest distance Laplacian eigenvalue $\partial_2(G) \geq n$ and $\partial_2(T) \geq 2n-1$, which confirm two conjectures in [2]; they also proved that the second largest distance signless Laplacian eigenvalue $q_2(T) \geq 2n-5$ (also see [6]), which confirms a conjecture in [3]. Das [6] obtained that for any connected unicyclic graph G, $q_2(G) \geq 2n-5$. Recently, Lin and Das [7] determined the lower bound of $q_2(G)$ for any connected n-vertex graph G in terms of the independence number α of G:

$$q_2(G) \geq \frac{3n + 2\alpha - 6 - \sqrt{n^2 - 4n\alpha + 4n + 12\alpha^2 - 16\alpha + 4}}{2}$$

with equality if and only if G is isomorphic to $K_{n-\alpha} \vee \alpha K_1$.

Inspired by the above result, we naturally want to ask: how about the lower bound of $q_2(G)$ or $\partial_2(G)$ of G in terms of other parameters. Furthermore, as we know, matching number is an important structure parameter in graph theory. Therefore, in this paper we investigate the relation between matching number and $\partial_2(G)$, obtaining the following conclusion.

Let $K_{m,n-m}$ be a complete bipartite graph with partitions $|X| = m$ and $|Y| = n-m$. Suppose any vertex $u \in Y$, then denote by $K_{m,n-m-1} + u$ the graph obtained from $K_{m,n-m}$ by removing the edges incident with u such that u is a pendant vertex.

Theorem 1.1. *Let G be a connected simple graph with order $n \geq 4$ and matching number $m(G) = m$. Then the following assertions hold.*

- (i) *If $m = \lfloor \frac{n}{2} \rfloor$, then $\partial_2(G) \geq n$ with equality holding if and only if G is K_n or $K_n - e$ with $e \in E(K_n)$ being an edge.*
- (ii) *If $1 \leq m \leq \lfloor \frac{n}{2} \rfloor - 1$, then $\partial_2(G) \geq 2n - m$, with equality holding if and only if G is a spanning subgraph of $K_m \vee \overline{K_{n-m}}$ and G contains $K_{m,n-m-1} + u$ as a spanning subgraph.*

Remark 1.2. The authors of [2] conjectured that for a tree G of order $n \geq 5$, $\partial_2(G) \geq 2n-1$ with equality if and only if G is the star S_n, which is proved in [16]. Since the