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Estimates of the discrete van Cittert deconvolution error in approximate deconvolution models of turbulence in bounded domains

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ABSTRACT

Recent turbulence models such as the Approximate Deconvolution Model (ADM) or the Leray-deconvolution model are derived from the Navier–Stokes equations using the van Cittert approximate deconvolution method. As a consequence, the numerical error in the approximation of the Navier–Stokes weak solution with discrete solutions of the above models is influenced also by the discrete deconvolution error $\mathbf{u} - D_N\overline{\mathbf{u}}^h$ caused by the approximate deconvolution method. Here $\mathbf{u}$ is the flow field, $\overline{\mathbf{u}}$ is its average and D_N is the N-th order van Cittert deconvolution operator.
It is therefore important to analyze the deconvolution error $\mathbf{u} - D_N\overline{\mathbf{u}}^h$ in terms of the mesh size h, the filter radius α and the order N of the deconvolution operators used in the computation.
This problem is investigated herein in the case of bounded domains and zero-Dirichlet boundary conditions. It is proved that on a sequence of quasiuniform meshes the L^2 norm of the discrete deconvolution error convergences to 0 in the order of $h^{k+1} + K^N h$ provided that the filter radius is in the order of the mesh size and the flow field has enough regularity. Here $K < 1$ is a parameter that depends on the ratio *(filter radius)/(mesh size)* (which is assumed constant) and k is the order of the FE velocity space. The rate of convergence of the H^1 norm of the error is also provided. The estimates are valid for the differential and Stokes filters. The result proves that higher order deconvolution operators decrease the deconvolution error and can be used to increase accuracy in approximate deconvolution models of flow problems.

1. Introduction

The van Cittert approximate deconvolution method, [5], has been used for flow problems starting with the work [1,42–44] of Stolz, Adams and Kleiser. The method has been later investigated and tested in many works such as [2–4,9,10,12,18,24,29,30,34–36,38–41] and used as a regularization tool to devise or improve flow models such as the approximate deconvolution turbulence models (ADM), [2,24,30,41], the Leray-deconvolution, [28,32], reduced ADM, [4,18], approximate deconvolution NS-alpha model, [9,37], the Vreman model investigated in [15], the time-relaxation model in [16]. It has also been used in first order hyperbolic systems [8,14] and MHD [3].

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When a weak solution $\mathbf{u}$ of the Navier–Stokes equations is approximated by finite element discrete solutions of such approximate deconvolution based turbulence models the resulting finite element error will contain a residual part which is dominated by the discrete deconvolution error $\mathbf{u}-D_N\overline{\mathbf{u}}^h$ of the van Cittert deconvolution method. Here $\overline{\mathbf{u}}^h$ is its average and D_N is the N-th order van Cittert deconvolution operator.

It is therefore important to analyze the discrete deconvolution error $\mathbf{u}-D_N\overline{\mathbf{u}}^h$ as it influences the overall rate of convergence of discretizations of turbulence models based on approximate deconvolution to the NSE.

Herein we investigate precisely this issue of convergence of the discrete deconvolution error $\mathbf{u}-D_N\overline{\mathbf{u}}^h$ with respect to the mesh size h, the filter radius α and the order N of the deconvolution operator $D_N\overline{\mathbf{u}}$. The analysis is done in the context of bounded domains and zero boundary conditions. The main result proved herein is Theorem 3.2 which we detail below.

For a given function $\mathbf{v}$, its continuous average $G\mathbf{v}=\overline{\mathbf{v}}$ is defined herein and in most papers cited above as the solution of the elliptic BVP

$$-\alpha^2\Delta\overline{\mathbf{v}}+\overline{\mathbf{v}}=\mathbf{v},$$

or the solution of the resolvent Stokes problem

$$\begin{aligned}-\alpha^2\Delta\overline{\mathbf{v}}+\overline{\mathbf{v}}+\nabla p &= \mathbf{v}\\ \nabla\cdot\overline{\mathbf{v}} &= 0.\end{aligned}$$

The parameter α is the filter radius. Herein we supplement the system above with the boundary condition $\mathbf{v}=0$ on the boundary. Other boundary conditions considered in the literature are periodic and Neumann boundary conditions.

In the continuous van Cittert deconvolution method, $\mathbf{v}$ is approximated by $D_N\overline{\mathbf{v}}$ where the continuous van Cittert deconvolution operators are defined as, [5],

$$D_N := \sum_{n=0}^{N}(I-G)^n, \quad N=0,1,2,\dots,$$

i.e. for $N=0,1,2$ the function $\mathbf{v}$ is approximated as

$$\begin{aligned}\mathbf{v} &\approx \overline{\mathbf{v}} && N=0\\ \mathbf{v} &\approx 2\overline{\mathbf{v}}-\overline{\overline{\mathbf{v}}} && N=1\\ \mathbf{v} &\approx 3\overline{\mathbf{v}}-3\overline{\overline{\mathbf{v}}}+\overline{\overline{\overline{\mathbf{v}}}} && N=2.\end{aligned}$$

For smooth $\mathbf{v}$ the continuous van Cittert deconvolution error $\mathbf{v}-D_N\overline{\mathbf{v}}$ takes the form, [29],

$$\mathbf{v}-D_N\overline{\mathbf{v}}=(-1)^{N+1}\alpha^{2N+2}(\Delta G)^{N+1}\mathbf{v}.$$

In the periodic setting and for smooth $\mathbf{v}$ it follows that

$$||\mathbf{v}-D_N\overline{\mathbf{v}}||=\mathcal{O}(\alpha^{2N+2}). \tag{1}$$

This property can be used to match the continuous deconvolution error convergence order with the orders of other terms arising in the computation.

In the case of bounded domains with zero-Dirichlet boundary conditions, which is the case considered herein, even for smooth functions $\mathbf{v}$ such a convergence order of the continuous deconvolution error as in (1) is attained only if $\mathbf{v}$ satisfies several additional conditions, see Layton, [26].

Herein we analyze instead the discrete deconvolution error $\mathbf{u}-D_N\overline{\mathbf{u}}^h$ in a bounded domain and for zero-Dirichlet boundary conditions and we prove that the order N of the discrete deconvolution operator D_N influences its size.

We show herein that if $\mathbf{v}$ belongs to the finite element space and α, h are kept fixed, then the L^2 norm of the discrete deconvolution error and its derivative behaves like

$$||\mathbf{v}-D_N\overline{\mathbf{v}}^h||+||\nabla\mathbf{v}-\nabla D_N\overline{\mathbf{v}}^h||\le\mathcal{O}(K^N), \tag{2}$$

when $N\to\infty$, where $K<1$ is a parameter that depends on the ratio $\dfrac{\alpha}{h}$.

This estimate can be used in Theorem 3.2 to prove that if $\mathbf{v}\in H^{k+1}(\Omega)\cap H_0^1(\Omega)$ is a divergence free vector field then the discrete deconvolution error satisfies

$$\begin{aligned}&||\mathbf{v}-D_N\overline{\mathbf{v}}^h||\le Ch^{k+1}|\mathbf{v}|_{H^{k+1}}+CK^N\alpha^2(\frac{\alpha^2}{h^2}+h+1)||\Delta\mathbf{v}||,\\ &||\nabla\mathbf{v}-\nabla D_N\overline{\mathbf{v}}^h||\le Ch^k|\mathbf{v}|_{H^{k+1}}+CK^N\frac{\alpha^2}{h}(\frac{\alpha^2}{h^2}+h+1)||\Delta\mathbf{v}||,\end{aligned}$$

where K depends on the ratio $\dfrac{\alpha}{h}$ (which is assumed constant), C is a general constant and k is the order of the FEM space.

In the last section we present several numerical tests to check formula (2). Our tests show that for fixed h,α the error behaves like $\mathcal{O}(K^N)$, where the constant K is determined in the computation as the upper limit of quotients of consecutive errors.

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