ARTICLE IN PRESS APNUM:3391

Applied Numerical Mathematics ••• (••••) •••–•••

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

QX factorization of centrosymmetric matrices

A. Steele, J. Yalim, B. Welfert *

School of Mathematical & Statistical Sciences, Arizona State University, Tempe, AZ 85287-1804, United States of America

ARTICLE INFO

Article history:
Received 8 June 2018
Received in revised form 27 June 2018
Accepted 27 June 2018
Available online xxxx

Keywords:
Centrosymmetry
QR factorization

ABSTRACT

We show how the factorization $A = QX$, introduced in Burnik (2015) [2], of a real centrosymmetric $m \times n$ matrix A into a centrosymmetric orthogonal $m \times m$ matrix Q and a centrosymmetric $m \times n$ matrix X with a double-cone structure can be directly obtained via standard QR factorizations of two matrices about half the size of A. Examples and a MATLAB code are included.

1. Introduction

A matrix $A \in \mathbb{R}^{m\times n}$ is centrosymmetric if and only if [6]

$$J_m A = A J_n,$$

where

$$J_m := \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \in \mathbb{R}^{m\times m},$$

and J_n is similarly defined. Premultiplying A by J_m reverses the ordering of its rows, while postmultiplying A by J_n reverses the ordering of its columns (the notation R – for "reflection" – is also often used for J, but avoided here to prevent confusion with an upper triangular matrix). Thus, a centrosymmetric matrix is symmetric about its center.

Centrosymmetric matrices occur in many signal processing applications [3]. Our own interest in this class of matrices stems from pseudospectral approximations of differential equations. In particular, second-order differentiation matrices based on collocation sets which are symmetric with respect to the origin are centrosymmetric.

Square centrosymmetric matrices can be easily block-diagonalized via an orthogonal similarity transformation [6, Th. 9, p. 714], a first step in the efficient full diagonalization or inversion of these matrices [1,4].

While much focus has been placed on this full diagonalization, very little work has been done on QR-like factorizations, which are often building blocks in diagonalization algorithms, of centrosymmetric matrices. Burnick [2] recently introduced a constructive algorithm to factor a centrosymmetric matrix $A \in \mathbb{R}^{m\times n}$ as a product of a centrosymmetric orthogonal matrix Q and a centrosymmetric matrix X with a double-cone structure. For example,

* Corresponding author.
E-mail addresses: ajsteele@asu.edu (A. Steele), yalim@asu.edu (J. Yalim), welfert@asu.edu (B. Welfert).

https://doi.org/10.1016/j.apnum.2018.06.006
0168-9274/

Please cite this article in press as: A. Steele et al., QX factorization of centrosymmetric matrices, Appl. Numer. Math. (2018), https://doi.org/10.1016/j.apnum.2018.06.006

$$A = \begin{bmatrix} -2 & 3 & -3 & -1 \\ 2 & 2 & 3 & 2 \\ 2 & 3 & 2 & 2 \\ -1 & -3 & 3 & -2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -4 & 2 & 2 & 1 \\ 2 & -1 & 4 & 2 \\ 2 & 4 & -1 & 2 \\ 1 & 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} 3 & -1 & 5 & 2 \\ & 2 & 1 & \\ & 1 & 2 & \\ 2 & 5 & -1 & 3 \end{bmatrix} = QX. \tag{1}$$

Any symmetric permutation of the columns of Q and corresponding rows of X preserves the centrosymmetric properties of Q and X, the orthogonality of Q, and the double-cone structure of X, so that the factorization (1) is not unique, even with non-negative diagonal coefficients of X. Interestingly, none of these permutations leads, in this example, to a matrix Q which belongs to the connected component containing the 4×4 identity matrix, as characterized in [5, B.2].

Example 3.3 of [2] illustrates the factorization for a rectangular matrix A with odd dimensions m and n:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0.2 & 4 & 5 \\ 3 & -1 & 3 \\ 5 & 4 & 0.2 \\ -1 & 2 & 1 \end{bmatrix} = QX, \tag{2a}$$

with

$$Q \approx \begin{bmatrix} 0.192308 & -0.189023 & -0.59285 & 0.734054 & -0.192308 \\ -0.0741255 & 0.072705 & 0.243637 & 0.45732 & 0.848951 \\ 0.447015 & -0.459355 & -0.422304 & -0.459355 & 0.447015 \\ 0.848951 & 0.45732 & 0.243637 & 0.072705 & -0.0741255 \\ -0.192308 & 0.734054 & -0.59285 & -0.189023 & 0.192308 \end{bmatrix}, \tag{2b}$$

(corrected from Q in [2]) and

$$X \approx \begin{bmatrix} 5.95559 & 2.65229 & 0.755592 \\ & 3.66952 & \\ & & \\ & 3.66952 & \\ 0.755592 & 2.65229 & 5.95559 \end{bmatrix}. \tag{2c}$$

Centrosymmetry and orthogonality of Q also guarantee [2, Prop. 2.14]

$$Q^T J_m Q = Q^T Q J_m = J_m. \tag{3}$$

Matrices satisfying (3) are called perplectic [5, (2.8c)].

Burnick's algorithm for obtaining Q and X uses centrosymmetric Householder reflection matrices to centrosymmetrically zero-out appropriate coefficients of A, and closely follows the process at the core of the traditional Householder QR factorization. Instead, we show here how the standard QR factorizations of the two diagonal blocks in the generalization, equations (4) and (9), of the block-diagonalization exposed in [6, Th. 9, p. 714] to rectangular centrosymmetric matrices, can be leveraged to obtain an equivalent QX factorization, in a conceptually and practically simpler fashion.

2. Centrosymmetric QX

2.1. Even dimensions $(m, n) = (2p, 2q)$

The construction of the matrices Q and X is best explained first in the even dimensions case. An $m \times n$ centrosymmetric matrix A has the structure (compare [6, Th. 9(a)])

$$A = \begin{bmatrix} A_1 & A_2 J_q \\ J_p A_2 & J_p A_1 J_q \end{bmatrix} = U_p \begin{bmatrix} A_1 + A_2 & \\ & A_1 - A_2 \end{bmatrix} U_q^T, \tag{4}$$

with $A_1, A_2 \in \mathbb{R}^{p \times q}$ and

$$U_k = \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & I_k \\ J_k & -J_k \end{bmatrix} \in \mathbb{R}^{2k \times 2k}, \quad k = p, q. \tag{5}$$

Define orthogonal matrices $Q_\pm \in \mathbb{R}^{p \times p}$ and upper triangular matrices $R_\pm \in \mathbb{R}^{p \times q}$ from the two standard QR factorizations

$$A_1 \pm A_2 = Q_\pm R_\pm. \tag{6}$$